Résumé

Soit \( (P) \) un semi-groupe de Borel, et soit \( (S) \) l’inverse continu à droite d’une fonctionnelle additive continue \( (R) \). Soit \( \{Y_t\}_{t \in \mathbb{R}} \) un processus stationnaire à naissance et mort aléatoires, markovien de semi-groupe \( (P) \) sous la mesure de Kuznetsov \( Q \) associée à une mesure excessive. On définit, sous l’hypothèse que la mesure caractéristique \( \mu_x = \int_{I_{+}} \mu(dt) \) de \( (B) \) est purement excessive pour la \( \sigma \)-algèbre engendrée par \( \{P_s\}_{s \geq 0} \), une fonctionnelle additive pour \( \{Y_t\}_{t \in \mathbb{R}} \) en fonction de \( (B) \) et on étudie les lois des excursions associées à l’ensemble régénératif constitué des temps de discontinuité de l’inverse continu à droite \( (U) \) de cette fonctionnelle additive. Plus précisément, si on note par \( \{\Phi_t\} \) le processus \( \{Y_{t_i}\} \) et par \( H \) la \( \sigma \)-algèbre engendrée par \( H_t \) \( (t \in \mathbb{R}) \) où \( H_t \) est la \( Q \)-complétion de \( H^0 \), \( \{H^0_t\} \) étant la filtration naturelle de \( \{\Phi_t\} \), alors si \( T \) est un \( (H^0) \)-temps d’arrêt tel que \( U_T \neq U_T' \) est \( \Phi_T \neq \Phi_T' \), la loi conditionnelle de l’excursion chevauchant \( \{U_T, U_T'\} \) par rapport à \( H \) dépend uniquement de \( \Phi_T \) et de \( \Phi_T' \). Les lois conditionnelles des couples d’excursions ont été également étudiées. MSC: 60J25; 60J40; 60J55.

Mot clés: Standard process; Predictable process; Excursion; Additive functional; Conditional law; Exit measure; Kuznetsov process.

Abstract

Let \( (P) \) be a right borel semigroup and let \( (S) \) the right inverse of a continuous additive functional \( (R) \). Let \( \{Y_t\}_{t \in \mathbb{R}} \) be a right stationary process with random birth and death, Markov with semi group \( (P) \) under the Kuznetsov measure \( Q \) associated to an excessive measure. We define, under the assumption that the characteristic measure \( \mu_x = \int_{I_{+}} \mu(dt) \) of \( (B) \) is purey excessive for the semigroup \( (P) \), an additive functional for \( \{Y_t\}_{t \in \mathbb{R}} \) in terms of \( (B) \) and we study the laws of excursions associated to the regenerative set which consists in times of discontinuity of the right inverse \( (U) \) of this additive functional. More precisely, if we note by \( \{\Phi_t\} \) the process \( \{Y_{t_i}\} \) and by \( H \) the \( \sigma \)-algebra generated by \( H_t \) \( (t \in \mathbb{R}) \) where \( H_t \) is the \( Q \)-completion of \( H^0_t \), \( \{H^0_t\} \) is the natural filtration of \( \{\Phi_t\} \), then if \( T \) is a \( \{H^0_t\} \)-stopping time such that \( U_T \neq U_T' \) and \( \Phi_T \neq \Phi_T' \), the conditional law of the excursion straddling \( \{U_T, U_T'\} \) with respect to \( H \) depend only on \( \Phi_T \) and \( \Phi_T' \). Conditional laws of pairs of excursions are also considered.

Keywords: Standard process; Predictable process; Excursion; Additive functional; Conditional law; Exit measure; Kuznetsov process.

MSC: 60J25; 60J40; 60J55

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In 1985 Kaspi [6] constructs, for a standard process \( X \), an additive functional \( B \) associated to a regenerative system \( M \) and gives, under the classical duality hypothesis, the probability measures \( P_{x,y}^{\infty} \) allowing the law of excursions associated to \( B \) with respect to the \( \sigma \)-algebra \( K = \sigma(\{Z_t: t \in R_+\}) \), known to start at \( x \) and end at \( y \) \( (Z_t = X_{t_i}) \), where \( S_i = \inf \{u : B_u > t\} \).
It was given in [2], without duality, the measures $P^{x,y}$ in terms of the $\left\{ F_{t}\right\}$-predictable exit measures for a regenerative system consisting of the closure of the set of times that the regular points of an arbitrary continuous additive functional are visited. The purpose of this paper is to give the conditional laws of pairs of excursions for a Markov process with random birth and death $\left\{ Y_{i}\right\}_{i\in \mathbb{R}}$ having the same semigroup as $X$. To this respect, we define an "additive functional" for $\left\{ Y_{i}\right\}_{i\in \mathbb{R}}$ and we extend this result to $\left\{ Y_{i}\right\}_{i\in \mathbb{R}}$. Laws of pairs of excursions for $\left\{ Y_{i}\right\}_{i\in \mathbb{R}}$ are discussed.

**Preliminaries and notations**

Let $\left\{ \Omega,F,F_{t},X_{t},\theta_{t},P^{x}\right\}$ be the canonical realization for a borel standard semigroup $\left\{ P_{t}\right\}$. We assume that the state space $E$ is lusinian, and we note by $E$ its $\sigma$-algebra of borel sets. The cemetery point $\delta$ is absorbent and outside of $E$. Let $\left\{ B_{t}\right\}$ be a continuous additive functional and let $R$ be the perfect exact terminal time $\inf\left\{ u:B_{u}>0\right\}$. We note by $C=\left\{ x:P^{x}(R=0)=1\right\}$ the fine support of $\left\{ B_{t}\right\}$.

Let $F^{*}$ be the universal completion of the $\sigma$-algebra $\sigma\left\{ X_{t}:t\in \mathbb{R}_{+}\right\}$. We consider the random homogeneous set $M=\left\{ t+R\circ \theta_{t}:t\in \mathbb{R}_{+}\right\}$, and its family of $\left\{ F_{t}\right\}$-predictable exit measures $\left\{ P_{0}^{x}\right\}_{x\in E,\omega}$ (we assume that $R$ is $F^{*}$-measurable), where $D_{t}$ is the random variable $\inf\left\{ s>t:s\in M\right\}$. Note that if $S_{\tau}\neq S_{\tau}^{*}$, then $D_{\tau}^{*}=S_{\tau}$, hence the excursion associated to $t$ is defined by:

$$c(\omega)(s)=k_{t}^{*}(\omega)(s)=\begin{cases} X_{S_{\tau}^{*}}(\omega) & \text{if } s<S_{\tau}(\omega)-S_{\tau}^{*}(\omega) \\ \delta & \text{if } s\geq S_{\tau}(\omega)-S_{\tau}^{*}(\omega) \end{cases}$$

where $k_{t}^{*}$ killing operator at $t$ defined by:

$$k_{t}(\omega)(s)=\omega(s) \text{ if } s<t \text{ and } \delta \text{ if } s\geq t.$$ We consider for $(x,y)\in E\times E$ such that $x\neq y$, the measures $P^{x,y}$ on $\left\{ \Omega,F^{*}\right\}$ "defined by":

$$P^{x,y}=H^{x}(k_{R}^{-}\in .X_{R}=y) \text{ where } H^{x}=\begin{cases} P^{x} & (.;X_{R} \neq x) \end{cases}$$

Since $\left\{ \Omega,F^{0}\right\}$ is an U-space, and according to a classical lemma of Doob the measures $P^{x,y}$ can be chosen measurable for the pair $(x,y)$.

We associate to the right inverse $S_{t}=\left\{ u:B_{u}>t\right\}$ of $\left\{ B_{t}\right\}$, the following notations:

$$Z_{t}=X_{S_{t}},M_{t}=F_{S_{t}} \text{ and } \theta_{t}=F_{S_{t}}.$$ It is well known that the process $Z=\left\{ \Omega,F,M_{t},Z_{t},\theta_{t},P^{x}\right\}$ is strong Markov with semigroup $\left\{ P_{t}\right\}=\left\{ P_{S_{t}}\right\}$ and takes values on $\left[ C,C\cap E^{*}\right]$ (cf. Jacobs [5]).

Let $\left\{ K_{t}\right\}_{t\in \mathbb{R}_{+}}$ be the filtration, where $K_{t}$ is the intersection of the $P^{x}$-completions of the $\sigma$-algebra $K_{t}^{0}$, where $\pi$ is in the set of all the bounded measures on $E$; $\left\{ K_{t}^{0}\right\}$ is the natural filtration of the process $\left\{ Z_{t}\right\}$. It was shown in [2] that if $T$ is a finite $\left\{ K_{t}\right\}$-stopping time such that $S_{T}^{*}\neq S_{T}$ a.s., then we have:

$$\left. F_{\left(S_{T}^{*}\right)}^{-}\right| = K_{T} \tag{1}$$

and that if $S_{\tau}\neq S_{T}$ and $Z_{\tau}\neq Z_{T}$ a.s., then

$$P\left(f\left| \left| K_{T}\right| \right.\right)=P^{z_{\tau},z_{T}}\left( f \right) \tag{2}$$

for all positive and $F^{*}$-measurable function $f$, where

$$P\left(A\right)=\int P^{x}(A)\mu(dx)$$

$(\mu)$ is an arbitrary law on $E$).

Note that the formula (2) was proved by Kaspi [8] under the duality hypothesis.

**Excursions of Kuznetsov processes**

Let $W$ be the set of applications $w:R\mapsto E\cup\left\{ \delta\right\}$ which satisfies the following properties: there exits an open subset of $R$ on which $w$ is $E$-valued right -continuous with left limits and out which equals $\delta$. We note by $\left\{ \tilde{Y}_{t}\right\}$ the coordinate process on $W$. 

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Let \((G^0)_{t \in \mathbb{R}}\) be the natural filtration of \((Y_t)_{t \in \mathbb{R}}\) and let 
\(G^0 = \bigcup_{t \in \mathbb{R}} V G^0 \). Then the birth and the death times of 
\((Y_t)_{t \in \mathbb{R}}\) are respectively:

\[
\alpha = \inf \{ t \in \mathbb{R} : Y_t \in E \} \quad (\inf \mathcal{O} = +\infty), \\
\beta = \sup \{ t \in \mathbb{R} : Y_t \in E \} \quad (\sup \mathcal{O} = -\infty).
\]

We define the families of operators on \(W\) by:

\[
\tau_t : W \rightarrow \Omega \text{ such that } \tau_t(w) = w(s+t) \text{ for } s \in \mathbb{R}, t \in \mathbb{R}, \\
\sigma_t : W \rightarrow W \text{ such that } \sigma_t(w) = w(s+t) \text{ for } s, t \in \mathbb{R}.
\]

Note that \(X_s \circ \tau_t = Y_{t+s}\) on \(\{ Y_t \in E \}\) and 
\(\sigma_t \circ \sigma_u = \sigma_{t+u}\) for \(t, u \in \mathbb{R}\), \(s \in \mathbb{R}\). Let \(\eta\) be an 
eccessive measure with respect to \((P_t)\) and let \(Q\) be the Kuznetsov measure on \(W\) that corresponds to 
\(\eta_t(P_t)\) (cf. \([8],[10]\)). We note by \(G_t\) and \(G\) the \(Q\)-
completions of \(G^0\) and \(G^0\), and we assume that the 
semigroup \((P_t)\) satisfies "les hypothèses droites de Meyer". It follows by 
\([10]\) that the process 
\(Y = (W, G, G, (Y_t)_{t \in \mathbb{R}}, \tau_t, \alpha, \beta, Q)\) is stationary 
(i.e. \(\sigma_t(Q) = Q\)) and strong Markov with 
semigroup \((P_t)\).

For the generalization of formula (2), we consider 
the additive functionals \(B\) and \(S\) given in the previous 
section. We also note by \(B\) the random measure on \(W\), 
carried by \([\alpha, \beta]\) such that:

\[
B_s \circ \tau_t = B_{s+t}, \text{ on } \{ Y_t \in E \} \text{ for all } s > 0 \text{ and } t \in R.
\]

We assume that the characteristic measure 
\(\nu_B = Q \int_{Y_t \in E} B(dt)\) of \(B\) is purely excessive for the 
semigroup \((P_t)\) (i.e. \(\int P_t f(x) \nu_B(dx) \rightarrow 0\) as \(t \rightarrow \infty\) 
if \(\nu_B(f) < \infty\)). It was shown in \([7]\) that \(Q\) a.e. 
\(B[\alpha, t] < \infty\) for all \(t > \alpha\).

Let \(V_t = V_t(B, \alpha, t)\) be the nondecreasing process defined 
on \(W\) by:

\[
V_t = \alpha + B[\alpha, t] \text{ on } \{ \alpha < t \} \text{ and } V_t = \alpha \text{ on } \{ t \leq \alpha \},
\]

and let \(U_t\) be the right-continuous inverse of 
\(\{V_t\}_{t \in \mathbb{R}}\) that is:

\[
U_t = \inf \{ u > \alpha : V_u > t \}
\]

We also note by \(M\) the closed random subset of \([\alpha, \beta]\) 
defined by: \(M = \alpha < t < \beta \cup \{ t + R \circ \tau_t \}\) which verifies 
the following property of homogeneity (cf. \([4]\)):

\[
(M - t) \cap [0, \infty) = M \circ \tau_t, \text{ on } \{ Y_t \in E \}
\]

Remark:
1) If \(\alpha = -\infty\), \(\{ u > \alpha : V_u > t \} = \emptyset\) and \(U_t = +\infty\), then 
\(\alpha > -\infty\) on \(\{ \alpha < U_t < \beta \}\).
2) \(U_t = \alpha\) on \(\{ t \leq \alpha \}\).

For \(t \in \mathbb{R}\), let 
\(\Phi_t = Y_t, G_t = G_{U_t}, \tau_t = \tau_{U_t}, H_t = \sigma(\Phi_t : u \leq t)\) 
and \(H^0 = \sigma(H_t : t \in \mathbb{R})\). We note by \(H_t\) (resp. \(H^0\)) the 
\(Q\)-completion of \(H_t\) (resp. \(H^0\)). Note that for all the 
following formulas, the \(\sigma\)-finiteness of \(Q\) is guaranteed 
by the argument used in \([1]\). It is not hard to show that 
\(\{\Phi_t\}\) has the same properties as \(\{Z_t\}\) and the following 
result hold.

Proposition:
1) The process \(\{U_t\}\) is right continuous, has left limits, and 
satisfies \(U_t = U_{t+}\) for all \(t \geq \beta\) \(Q\) a.e.,
2) \(\{U_t\}\) is \((G_t)\)-adapted.
3) For all \(t \in \mathbb{R}\) and \(s > 0\) we have:
a) \(U_t = \alpha + S_{t-s} \circ \tau_{t-s}\) on \(\{ -\infty < \alpha < t \}\)

b) \(V_{t+s} = V_t + B_s \circ \tau_t\) on \(\{ Y_t \in E \}\) and 
\(U_{t+s} = U_t + S_s \circ \tau_t\) on \(\{ \alpha < U_t < \beta \}\).
4) On \(\{U_t \neq U_{t+}\}\), \(U_t, U_{t+}\) is a contiguous interval 
of \(M\).

If \(U_t \neq U_{t+}\), let \(E_t\) be the excursion associated to 
\(B\) defined by:

\[
E_t(w)(s) = \begin{cases} 
Y_{\tau_{V_t}(w)} & \text{if } 0 \leq s < U_t(w) - U_{t-}(w) \\
\delta & \text{if } s \geq U_t(w) - U_{t-}(w)
\end{cases}
\]

According to the previous proposition, the process 
\(\{V_t\}_{t \in \mathbb{R}}\) has got the same role as \(B\) for the process
Theorem 1:
1) The process $\Phi = (W, \Phi_t, G, \mathcal{G}_t, \tau, Q)$ is strong Markov in the sense that for all $(\mathcal{G}_t)$-stopping time $T$ and $s > 0$:
$$Q(f(\Phi_{t+s}) | \mathcal{G}_T) = \mathbb{P}(f(\Phi_t) | \mathcal{F}_t)$$
for all functions positive and $\mathcal{F}_t$-measurable $f$.

2) Assume that $T_t$ is a finite $(H_\xi)$-stopping time such that $U_{T_t} \neq U_{T_t}$ and $\Phi_{T_t} \neq \Phi_{T_t}$, $Q$ a.e.. Then we have:
$$Q(F | E_{T_t}) = P^{\Phi_{T}, \Phi_{T}}(F)$$
for all $F \geq 0$, $F^*$-measurable.

Proof: If $T \equiv t$ is constant, the formula (3) follows from the Markov property of the process $(Y_t)$ at time $U_t$ and the fact that $\Phi_{T_t} = Z_t \Phi_{T_t}$ and $\tau_{T_t} = \Theta_t \tau_{T_t}$ on $\{\alpha < U_t < \beta\}$. This formula is also true for $T_t$ instead of $T$, where $(T_t)$ is the decreasing dyadic approximation of $T$, which extends for a general $T$ by the right continuity of the processes $(\Phi_t)$, $(U_t)$ and $(\tau_t)$. The formula (4) is argued in the same manner as (2) by using the formula (30) of [1].

Conditional laws of pairs of excursions
We consider now the time-reversed process $(\tilde{Y}_t)_{t \in \mathbb{R}} = (Y_{-t})_{t \in \mathbb{R}}$. It is an $E$-valued right-continuous process with left limits on $]-\beta, -\alpha]$ and is equal to $\delta$ outside of $[-\alpha, -\beta]$. As in [1] and [10], we assume that $(\tilde{Y}_t)$ is also Markov with respect to another standard semigroup $(\tilde{P}_t)$ satisfying "les hypothèses droites de Meyer", which implies the strong Markov property and the existence of exit systems. The measure
$$\tilde{\eta}(B) = Q(\tilde{Y}_t \in B; \tilde{\alpha} < t < \tilde{\beta})$$
is $(\tilde{P}_t)$-excessive and the stationarity of $(\tilde{Y}_t)$ is guaranteed. Let $\hat{T}, G, B, \hat{S}, \hat{V}_t, \hat{U}_t$ and $\hat{E}_t$ be the analogues of $T, G_t, B_t, S_t, V_t, U_t$, and $E_t$ corresponding to $(\tilde{Y}_t)$. As previously we assume that $Q$ a.e. $\hat{B} [\alpha, t] < \infty$ for all $t > \alpha$. For the process $(\tilde{Y}_t) = (\tilde{Y}_{\tilde{U}_t})$ and the random subset
$$\tilde{M} = \tilde{\alpha} < t < \tilde{\beta} \cup \{t + R \cdot \tilde{\tau}_t\}$$
of $[\alpha, \beta]$, we have the analogous of theorem 1. In particular if we design by $\tilde{H}$ and $\tilde{H}$, the $Q$ completions of $\Sigma(\tilde{\Psi}_u : u \in \mathbb{R})$ and $\Sigma(\tilde{\Psi}_u : u \leq \tilde{t})$, respectively, and by $\tilde{P}$ the measure defined as $P^\tilde{\gamma}$ in terms of the exit measures $\tilde{P}^{x}$ of $\tilde{M}$ for the canonical realization of $(\tilde{P}_t)$, we have the following formula:
$$Q(F | \tilde{E}_{T_t}) = \tilde{P}^{\tilde{\Psi}_u, \tilde{\Psi}_{T_t}}(F)$$
for all finite $(\tilde{H}_t)$-stopping time $T_t$ such that $\tilde{U}_{T_t} \neq \tilde{U}_{T_t}$ and $\tilde{\Psi}_{T_t} \neq \tilde{\Psi}_{T_t}$, $Q$ a.e., and for all positive function $F^*$-measurable $F$.

For the following theorem which gives the conditional law of pairs of excursions, we consider the family of probability measures
$$Q^{x,y,z,\mu} = P^{x,y} \otimes \tilde{P}^{z,\mu}.$$

Theorem 2:
Let $T_t$ (resp. $T_\mu$) as in (4) (resp. (5)). We assume that the following hypotheses are satisfied:
1) $\sigma(\tilde{U}_{X_t}) \cap \Lambda \subset \tilde{H}_{T_t}$
2) $\sigma(\tilde{U}_{T_t}) \cap \Lambda \subset \tilde{H}_{T_t}$,
where $\Lambda = \{\alpha < -\tilde{U}_{X_t} \leq U_{X_t} < \beta\}$. Then we have the following formula:
$$Q(H | E_{T_t}, E_{\tilde{T}_t}) = Q^{\Phi_{T_t}, \Phi_{\tilde{T}_t}, \tilde{\Psi}_{T_t}, \tilde{\Psi}_{\tilde{T}_t}}(H)$$
on $\Lambda$ for all positive and $F^*$-measurable function $H$.

Proof:
We have to prove that:
$$Q(F | E_{T_t}, E_{\tilde{T}_t}) = Q^{\Phi_{T_t}, \Phi_{\tilde{T}_t}}(F) \tilde{P}^{\tilde{\Psi}_{T_t}, \tilde{\Psi}_{\tilde{T}_t}}(F)$$
(7)
for all $F, F^*$ positive functions and $F^*$-measurable, and for all positive random variable $Z$, $H \cap \varpi$-measurable.

Since $\tau_{U_2} - \tau_{\varpi_2} = \theta_{U_1} \circ \tau_{U_2} - \tau_{U_2}$ on $\Lambda$, and since $\tau_{U_2}$ is $G_{\varpi_2} - \text{measurable}$ and $G_{\varpi_2} = H_{\varpi_2}$, then $\sigma(G_{\varpi_2}) \cap \Lambda \subset H_{\varpi_2}$, where $D_t = \inf \{ s > t : s \in M \}$ on $W$, for $t \in R$. By using the same argument, we prove that $\sigma(G_{\varpi_2}) \cap \Lambda \subset H_{\varpi_2}$, and $\sigma(G_{\varpi_2}) \cap \Lambda \subset H_{\varpi_2}$.

The Markov property at time $T_1$ and the formula (5) implied that for all positive random variable $Z_{\varpi_1}$, $H_{\varpi_1}$ -measurable and for all positive and $F^* - \text{measurable function } \varphi$:

$$Q(F|E_{T_1})F(E_{T_1})Z_\varpi(\tau_{\varpi_1})I_\Lambda = Q(P^{\varphi_{\tau_{\varpi_1}}}(F)^F(E_{T_1})Z_\varpi(\tau_{\varpi_1})I_\Lambda)$$

and according to the fact that $H$ is generated by $H_{\varpi_1}$ and $\tau_{\varpi_1}$, we have:

$$Q(F|E_{T_1})F(E_{T_1})Z_\varpi = Q(P^{\varphi_{\tau_{\varpi_1}}}(F)^F(E_{T_1})Z_\varpi)$$

The formula (7) follows by using formulas (8) and (5).

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