FLEXURAL VIBRATION OF A MULTI-DISK ROTOR WITH DIFFERENT BOUNDARY CONDITIONS

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Résumé
Dans cet article, les vibrations de flexion d'un rotor multi-disques sont considérées. L'analyse est effectuée pour le rotor avec des conditions aux limites appuyée-appuyée et appuyée-libre.

Celle-ci est faite dans le cas dissymétrique non amorti où les vitesses critiques et les réponses aux forces synchrones et asynchrones sont déterminées et comparées.

Mots clés : Vibrations, rotor dynamiques, rotor flexible, conditions aux limites, vitesses critiques.

Abstract
In this paper the flexural vibrations of a multi-disk rotor is considered. The analysis is made for the rotor when it is simply-supported and when it is free-supported.

This is done for the asymmetric undamped case where critical speeds and responses to synchronous and asynchronous forces are determined and compared.

Key words : Vibrations, rotordynamics, flexible rotor, boundary conditions, critical speed.

ملخص
في هذا المنشور تعتبر اهتزازات الألتواء لدوار متعدد الأقراص. يجري التحليل للدوار عندما يكون لديه دعم بسيط وعندما يكون لديه دعم بسيط-حر.

هذا يتم في الحالة غير المخمدة و غير المنتظرة حيث تحدد وتقارن السرعات الحرجة و الاستجابات للقوى المتزامنة وغير المتزامنة.

كلمات مفتاحية : الاهتزازات، الديناميكا الدوار، الدوار المرن، الشروط الحدودية، السرعة الحرجة.
1. INTRODUCTION

Rotating machinery, such as turbines, pumps, generators, and fans, play an important role in many different industries where they are considered among the masterpieces in the mechanisms [1,2]. Unfortunately, they are sources of vibrations that involve the phenomena of the fatigue of their materials as well as some bad comfort qualities, in addition to the resonance phenomenon that leads to disasters if it is not avoided. To ensure a good running it is, therefore, necessary to get a precise knowledge of the vibratory behavior of the rotating parts. The main point of the problem lies in the determination of the critical speeds.

In a previous work [3,4], the study on the vibration dynamic behavior of a flexible mono rotor in the case where it is symmetric was considered. The model chosen was of Lalanne and Ferraris [5] with different boundary conditions. The obtained results led to markedly different critical speeds and modes of vibration. In this work the same method has been followed but in an asymmetric case and for a rotor with different number of discs.

2. EQUATIONS OF MOTION

The model of rotor considered in this work is represented in figure 1. The study of its flexural vibrations is made for the cases where it is simply-supported and free-simply supported to see the effect of the boundary conditions on the modes of vibration and the critical rotating speeds. For a better comparison, the work is made for the rotor when it is with just with one disc, with two discs and with three discs.

The characteristics of the rotor, with the positions of the discs and bearings taken from the origin the inertial frame, are given as follows:

Shaft: Length \( L = 1.3 \) m, cross section radius \( r_s = 0.05 \) m, density \( \rho = 7800 \text{kg/m}^3 \), Young’s modulus \( E = 2.10^{11} \text{N/m}^2 \) and Poisson’s coefficient \( \nu = 0.3 \).

Disc1: Inner radius \( r_1 = 0.05 \) m, outer radius \( r_1 = 0.12 \) m, thickness \( h_1 = 0.05 \) m, density \( \rho = 7800 \text{kg/m}^3 \) and position \( L_1 = 0.45 \) m.

Disc2: Inner radius \( r_2 = 0.05 \) m, outer radius \( r_2 = 0.20 \) m, thickness \( h_2 = 0.05 \) m, density \( \rho = 7800 \text{kg/m}^3 \) and position \( L_2 = 0.65 \) m.

Disc3: Inner radius \( r_3 = 0.05 \) m, outer radius \( r_3 = 0.20 \) m, thickness \( h_3 = 0.06 \) m, density \( \rho = 7800 \text{kg/m}^3 \) and position \( L_3 = 0.85 \) m.

Bearing (1): Position \( L_1 = 0.2 \) m, \( k_{x1} = 5 \times 10^7 \text{N/m} \) and \( k_{z1} = 7 \times 10^7 \text{N/m} \).

Bearing (2): Position \( L_2 = 1.1 \) m, \( k_{x2} = 5 \times 10^7 \text{N/m} \) and \( k_{z2} = 7 \times 10^7 \text{N/m} \).

Mass imbalance (1): \( m_{01} = 8.8.10^{-2} \) kg and distance from the shaft axis \( d_1 = 0.12 \) m.

Mass imbalance (2): \( m_{023} = 3.34.10^{-4} \) kg and distance from the shaft axis \( d_2 = 0.20 \) m.

From these data we obtain the physical quantities:

Shaft: Cross-section \( s = 7.85.10^{-4} \text{m}^2 \) and diametral moment of inertia \( I_1 = 4.906.10^{-4} \text{m}^4 \).

Disc1: Mass \( M_{d1} = 14.57 \) kg and moments of inertia \( I_{d1} = 0.06459 \) kg m, \( I_{d1} = 0.123 \) kg m².

Disc2: Mass \( M_{d2} = 45.92 \) kg and moments of inertia \( I_{d2} = 0.497 \) kg m², \( I_{d2} = 0.9758 \) kg m².

Disc3: Mass \( M_{d3} = 55.107 \) kg and moments of inertia \( I_{d3} = 0.602 \) kg m², \( I_{d3} = 1.171 \) kg m².

Figure 1 : Considered Rotating Model

The analysis of the flexural vibrations of the rotor is carried out by modeling based on the Rayleigh-Ritz method which is characterized by the substitution by approximation functions of the displacements \( u \) and \( w \) in the \( x \) and \( z \) directions respectively:

\[ u(y, t) = f(y)q_1(t) = f(y)q_1 \]  \( (1) \)

\[ w(y, t) = f(y)q_2(t) = f(y)q_2 \]  \( (2) \)

where \( q_1 \) and \( q_2 \) are the generalized independent coordinates and \( f(y) \) is the displacement function which is taken for the first mode of a beam in flexion with a constant cross-section. It is given by:

For simply-supported case:

\[ f(y) = B \left[ \sin \beta_n y \right] \]  \( (3) \)

where \( \beta_n L = \pi \).

For free-simply supported case,

\[ f(y) = B \left[ \sin \beta_n y + \alpha_n \sinh \beta_n y \right] \]  \( (4) \)

where: \( \alpha_n = \frac{\sin \beta_n L}{\sinh \beta_n L} \) and \( \beta_n L = 3.9266 \).

B is a constant (taken equal to 1).

Consequently the expressions of the kinetic and stain energies (T and U) can be obtained.

The total kinetic energy of the system is:

\[ T = T_s + T_d + T_b \]  \( (5) \)

Where \( T_s, T_d \) and \( T_b \) are the kinetic energy of the shaft, the masses unbalance and the discs respectively and which are given by [3]:
\[ T_s = \frac{1}{2} \rho s (\dot{q}_1^2 + \dot{q}_2^2) \int_0^L f^2(y) \, dy \]  
\[ + \frac{1}{2} \rho I (\dot{q}_1^2 + \dot{q}_2^2) \int_0^L g^2(y) \, dy \]  
\[ T_d = T_{d1} + T_{d2} + T_{d3} \]  
(7)

with:
\[ T_{d1} = T_{d1}^{ma} + T_{d1}^{rot} = \frac{1}{2} M_{d1} f^2(l_1)(\dot{q}_1^2 + \dot{q}_2^2) \]  
\[ + \frac{1}{2} I_{d1} g^2(l_1)(\dot{q}_1^2 + \dot{q}_2^2) + \frac{1}{2} I_{d1} \Omega^2 - I_{d1} g^2(l_1) \dot{q}_1 \dot{q}_2 \Omega \]  
\[ T_{d2} = T_{d2}^{ma} + T_{d2}^{rot} = \frac{1}{2} M_{d2} f^2(l_2)(\dot{q}_1^2 + \dot{q}_2^2) \]  
\[ + \frac{1}{2} I_{d2} g^2(l_2)(\dot{q}_1^2 + \dot{q}_2^2) + \frac{1}{2} I_{d2} \Omega^2 - I_{d2} g^2(l_2) \dot{q}_1 \dot{q}_2 \Omega \]  
\[ T_{d3} = T_{d3}^{ma} + T_{d3}^{rot} = \frac{1}{2} M_{d3} f^2(l_4)(\dot{q}_1^2 + \dot{q}_2^2) \]  
\[ + \frac{1}{2} I_{d3} g^2(l_4)(\dot{q}_1^2 + \dot{q}_2^2) + \frac{1}{2} I_{d3} \Omega^2 - I_{d3} g^2(l_4) \dot{q}_1 \dot{q}_2 \Omega \]  
\[ T_b = T_{b1} + T_{b2} \]  
(8)

with:
\[ T_{b1} = m_{b1} d_1 \Omega f(l_2)(\dot{q}_1 \cos \Omega t - \dot{q}_2 \sin \Omega t) \]  
and
\[ T_{b2} = m_{b2} d_2 \Omega f(l_4)(\dot{q}_1 \cos \Omega t - \dot{q}_2 \sin \Omega t) \]  

The total strain energy is that of the shaft and it is given by:
\[ U_a = \frac{E I}{2} \left( \frac{d^2 f(y)}{d^2 y} \right) \int_0^L h^2(y) \, dy \]  
(9)

where:
\[ h(y) = \frac{d^2 f(y)}{d^2 y} \]

The total virtual work due to the stiffness of the bearings is:
\[ \delta W = -k_s f^2(l_1) \delta q_1 - k_s f^2(l_2) \delta q_2 - k_s f^2(l_4) \delta q_1 - k_s f^2(l_4) \delta q_2 \]  
(10)

Following the procedure made by Lallane and Ferrari, the equations of motion are deducted using Lagrange equations for the dissymmetric model.

In the free-simply supported case,
- Rotor with one disc:
\[ 52.7006 \ddot{q}_1 - 0.8269 \Omega \dot{q}_1 + 86.216 \, 10^6 \dot{q}_1 = 9.789 \, 10^5 \Omega^2 \sin \Omega t \]
\[ 52.7006 \ddot{q}_1 - 0.8269 \Omega \dot{q}_1 + 99.4867 \, 10^6 \dot{q}_1 = 9.789 \, 10^5 \Omega^2 \cos \Omega t \]  
(11)

- Rotor with two discs:
\[ 85.14 \ddot{q}_1 - 2.91 \Omega \dot{q}_1 + 86.216 \, 10^6 \dot{q}_1 = 9.789 \, 10^5 \Omega^2 \sin \Omega t \]
\[ 85.14 \ddot{q}_1 + 2.91 \Omega \dot{q}_1 + 99.4867 \, 10^6 \dot{q}_1 = 9.789 \, 10^5 \Omega^2 \cos \Omega t \]  
(12)

- Rotor with three discs:
\[ 98.126 \ddot{q}_1 - 14.076 \Omega \dot{q}_1 + 86.216 \, 10^6 \dot{q}_1 = 3.7199 \, 10^5 \Omega^2 \sin \Omega t \]
\[ 98.126 \ddot{q}_1 + 14.076 \Omega \dot{q}_1 + 99.4867 \, 10^6 \dot{q}_1 = 3.7199 \, 10^5 \Omega^2 \cos \Omega t \]  
(13)

In the simply-simply supported case,
- Rotor with one disc:
\[ 46.754 \ddot{q}_1 - 0.4793 \Omega \dot{q}_1 + 43.358 \, 10^6 \dot{q}_1 = 9.3477 \, 10^5 \Omega^2 \sin \Omega t \]
\[ 46.754 \ddot{q}_1 + 0.4793 \Omega \dot{q}_1 + 52.015 \, 10^6 \dot{q}_1 = 9.3477 \, 10^5 \Omega^2 \cos \Omega t \]  
(14)

- Rotor with two discs:
\[ 111.0564 \ddot{q}_1 - 35.6207 \Omega \dot{q}_1 + 43.358 \, 10^6 \dot{q}_1 = 9.3477 \, 10^5 \Omega^2 \sin \Omega t \]
\[ 111.0564 \ddot{q}_1 + 35.6207 \Omega \dot{q}_1 + 52.015 \, 10^6 \dot{q}_1 = 9.3477 \, 10^5 \Omega^2 \cos \Omega t \]  
(15)

- Rotor with three discs:
\[ 155.0624 \ddot{q}_1 - 37.0897 \Omega \dot{q}_1 + 43.358 \, 10^6 \dot{q}_1 = 1.00174 \, 10^6 \Omega^2 \sin \Omega t \]
\[ 155.0624 \ddot{q}_1 + 37.0897 \Omega \dot{q}_1 + 52.015 \, 10^6 \dot{q}_1 = 1.00174 \, 10^6 \Omega^2 \cos \Omega t \]  
(16)

Any of the above systems of equations can be put in the form
\[ \begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & m \end{bmatrix} \begin{bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \\ \ddot{q}_3 \end{bmatrix} + \Omega \begin{bmatrix} 0 & -a & 0 \\ a & 0 & 0 \\ 0 & 0 & k_2 \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \end{bmatrix} + \Omega^2 \begin{bmatrix} \sin \Omega t \\ \cos \Omega t \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = C \begin{bmatrix} \sin \Omega t \\ \cos \Omega t \end{bmatrix} \]  
(17)

or in the matrix form:
\[ \begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & m \end{bmatrix} \begin{bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \\ \ddot{q}_3 \end{bmatrix} + \Omega \begin{bmatrix} 0 & -a & 0 \\ a & 0 & 0 \\ 0 & 0 & k_2 \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \end{bmatrix} = C \begin{bmatrix} \sin \Omega t \\ \cos \Omega t \end{bmatrix} \]  
(18)

3. **NATURAL FREQUENCIES AND EIGENMODES**

The natural frequencies are found by solving the homogeneous system of equations (17) i.e the system without second member. Since this one is linear the solutions have the following forms:
\[ \begin{bmatrix} \ddot{q}_{1h} \\ \ddot{q}_{2h} \end{bmatrix} = A_1 \cos(\omega t + \phi_1) \]  
(19)

These can be transformed in a complex form as:
\[ \begin{bmatrix} \ddot{q}_{1h} \\ \ddot{q}_{2h} \end{bmatrix} = A_1 \exp j \omega t \]  
(20)

with \( A_1 = A_1 \exp j \phi_1 \) and \( A_2 = A_2 \exp j \phi_2 \).
By substitution in the homogenous complex system we get:

\[
\begin{align*}
\mathbf{m}\ddot{\mathbf{q}}_{2h} - a\Omega^2 \mathbf{q}_{2h} + \mathbf{k}_1 \mathbf{q}_{2h} &= 0 \\
\mathbf{m}\ddot{\mathbf{q}}_{1h} + a\Omega^2 \mathbf{q}_{1h} - \mathbf{k}_2 \mathbf{q}_{1h} &= 0
\end{align*}
\]  

(21)

Or in the matrix form

\[
\begin{bmatrix}
k_1 - \rho \omega^2 & -ja\Omega\alpha \\
 ja\Omega\alpha & k_2 - \rho \omega^2
\end{bmatrix} \begin{bmatrix} \mathbf{A}_1 \\ \mathbf{A}_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}
\]

(22)

This represents a linear system of equations with two unknown \( \mathbf{A}_1 \) and \( \mathbf{A}_2 \) with a parameter \( \omega \) which is discussed for the non-trivial case that corresponds to \( \text{det} = 0 \).

That is:

\[
\text{det} \left[ \begin{bmatrix} k_1 - \rho \omega^2 & -ja\Omega\alpha \\
 ja\Omega\alpha & k_2 - \rho \omega^2 \end{bmatrix} \right] = 0
\]

(23)

or:

\[
m^2 \omega^2 (k_1 + m k_2 + a^2 \Omega^2) \omega^2 + k_1 k_2 = 0
\]

(24)

We remark that when \( \Omega = 0 \) (rotor at rest), the solutions of the (24) are equal to:

\[
\omega_{10} = \sqrt{k_1/m}
\]

(25)

and:

\[
\omega_{20} = \sqrt{k_2/m}
\]

(26)

On the other hand when \( \Omega \neq 0 \) (rotating rotor), the bi-squared equation has a positive discriminant which means that it has two positive real values for \( \omega \) given by:

\[
\omega_1 = \sqrt{\frac{\omega_{10}^2 + \omega_{20}^2 + a^2 \Omega^2}{2m^2} - \left( \frac{\omega_{10}^2 + \omega_{20}^2 + a^2 \Omega^2}{2m^2} \right)^2 - 2\omega_{10}^2 \omega_{20}^2}
\]

(27)

and:

\[
\omega_2 = \sqrt{\frac{\omega_{10}^2 + \omega_{20}^2 + a^2 \Omega^2}{2m^2} - \left( \frac{\omega_{10}^2 + \omega_{20}^2 + a^2 \Omega^2}{2m^2} \right)^2 - 2\omega_{10}^2 \omega_{20}^2}
\]

(28)

We can find that:

\[
\omega_1 < \omega_{10} < \omega_{20} < \omega_2
\]

(29)

There are then two modes of vibration for each of the generalized coordinates \( \mathbf{q}_{1h} \) and \( \mathbf{q}_{2h} \) corresponding to the two values \( \omega_{10} \) and \( \omega_2 \).

i) For the first mode (\( \omega = \omega_{10} \)), we have:

\[
\frac{\mathbf{A}_{11}}{\mathbf{A}_{21}} = \frac{ja\Omega \omega_{10}}{k_1 - \rho \omega_{10}^2} = \frac{ja\Omega \omega_{10}}{m(\omega_{10}^2 - \omega_{10}^2)} = j
\]

(30)

That gives: \( \mathbf{A}_{11} = \mathbf{A}_{21} \) and \( \varphi_{21} = \varphi_{11} - \pi/2 \)

Hence:

\[
\mathbf{q}_{1h1} = \mathbf{A}_{11} \cos(\omega_{10} t + \varphi_{11})
\]

and:

\[
\mathbf{q}_{2h1} = \mathbf{A}_{21} \cos(\omega_{10} t + \varphi_{21}) = \mathbf{A}_{11} \sin(\omega_{10} t + \varphi_{11})
\]

(31)

Then, \( \mathbf{u}(y, t) = \mathbf{q}_{1h1} f(y) = \mathbf{A}_{11} \cos(\omega_{10} t + \varphi_{11}) [\sin \beta_y y + \alpha_y \sinh \beta_y y] \)

(32)

and:

\[
\mathbf{w}(y, t) = \mathbf{q}_{2h1} f(y) = \mathbf{A}_{11} \sin(\omega_{10} t + \varphi_{11}) [\sin \beta_y y + \alpha_y \sinh \beta_y y]
\]

(33)

ii) For the second mode (\( \omega = \omega_2 \)), we have:

\[
\frac{\mathbf{A}_{12}}{\mathbf{A}_{22}} = \frac{ja\Omega \omega_2}{k_1 - \rho \omega_2^2} = \frac{ja\Omega \omega_2}{m(\omega_{20}^2 - \omega_2^2)} = -j
\]

(34)

That gives:

\( \mathbf{A}_{12} = \mathbf{A}_{22} \) et \( \varphi_{22} = \varphi_{12} + \pi/2 \)

Hence:

\[
\mathbf{q}_{1h2} = \mathbf{A}_{12} \cos(\omega_2 t + \varphi_{12})
\]

and:

\[
\mathbf{q}_{2h2} = \mathbf{A}_{22} \cos(\omega_2 t + \varphi_{22}) = -\mathbf{A}_{12} \sin(\omega_2 t + \varphi_{12})
\]

(35)

Then:

\[
\mathbf{u}(y, t) = \mathbf{q}_{1h2} f(y) = \mathbf{A}_{12} \cos(\omega_2 t + \varphi_{12}) [\sin \beta_y y + \alpha_y \sinh \beta_y y]
\]

(36)

And:

\[
\mathbf{w}(y, t) = \mathbf{q}_{2h2} f(y) = \mathbf{A}_{22} \cos(\omega_2 t + \varphi_{22}) [\sin \beta_y y + \alpha_y \sinh \beta_y y]
\]

(37)

As \( \mathbf{A}_1 \) is different from \( \mathbf{A}_2 \), the orbits described by the rotor are ellipses with an inverse precession for the first mode and a direct precession for second one.

4. Campbell Diagram

The characteristic equation for each considered case is obtained from the corresponding system of equations (11) to (16). It allows having the frequencies at rest and the frequencies in rotation.

In Campbell Diagram (Figure 2 and Figure 3), the functions \( \omega_{10} = \omega_{10}(\Omega) \) et \( \omega_{20} = \omega_{20}(\Omega) \) are represented cut by the straight lines \( \omega = \Omega \) (the synchronous case) and \( \omega = \omega_0 \) (an asynchronic case with \( s = 0.5 \)) to get the intersection points A and B for the first and, C and D for the second.

The values of the frequencies corresponding for these points are obtained using \( \omega = \omega_0 \) in (24), that is:

\[
s^2(\omega_{10}^2 - \omega_{20}^2)\Omega^4 - m(k_1 + k_2)s^2\Omega^2 + k_1 k_2 = 0
\]

(38)

From this we find the critical value \( \omega_c \) for the cases \( s = 1 \) and \( s = 0.5 \).

In the simply-simply supported case, - Rotor with one disc:

\[
\omega_0 = \sqrt{\frac{k_1}{m}} = 0.962,997 \text{ rd/s}
\]

and \( \omega_0 = \sqrt{\frac{k_1}{m}} = 10.577,763 \text{ rd/s} \)

(40)
\[ \omega_2 = \sqrt{0.199 \times 10^8 + 2.456 \times 10^7 \Omega^2 + \left(0.199 \times 10^8 + 2.456 \times 10^7 \Omega^2\right)^2 - 1.031 \times 10^7} \tag{40} \]

A: \( \Omega_{c1} = 958 \text{ rd/s} \); B: \( \Omega_{c2} = 0.954 \text{ rd/s} \),
C: \( \Omega_{c1} = 1812.974 \text{ rd/s} \); D: \( \Omega_{c2} = 2192.32 \text{ rd/s} \).

- Rotor with two discs:
  \[ \omega^4 - (8.587 \times 10^5 + 12.68 \times 10^3 \Omega^2)\omega^2 + 2.255 \times 10^{15} = 0 \tag{42} \]
  \[ \omega_0 = \sqrt{k_i / m} = 624.831 \text{ rd/s} \] and \( \omega_{10} = \sqrt{k_i / m} = 684.372 \text{ rd/s} \) \tag{43}

\[
\begin{align*}
\omega_1 &= \sqrt{4.2938 \times 10^5 + 5.1438 \times 10^3 \Omega^2 - \sqrt{4.2938 \times 10^5 + 5.1438 \times 10^3 \Omega^2}} - 1.828 \times 10^5 \\
\omega_2 &= \sqrt{4.2938 \times 10^5 + 5.1438 \times 10^3 \Omega^2 + \sqrt{4.2938 \times 10^5 + 5.1438 \times 10^3 \Omega^2}} - 1.828 \times 10^5 
\end{align*}
\tag{44} \]

**Figure 2(a)**: Rotor with one disc, simply-simply supported case

**Figure 2(b)**: Rotor with two discs, simply-simply supported case

**Figure 2(c)**: Rotor with three discs, simply-simply supported case

A: \( \Omega_{c1} = 565.356 \text{ rd/s} \); B: \( \Omega_{c2} = 798.5135 \text{ rd/s} \),
C: \( \Omega_{c1} = 1017.45 \text{ rd/s} \); D: \( \Omega_{c2} = 2191.33 \text{ rd/s} \).

- Rotor with three discs:
  \[ \omega^4 - (6.1506 \times 10^4 + 13.756 \times 10^2 \Omega^2)\omega^2 + 2.255 \times 10^{15} = 0 \tag{45} \]
  \[ \omega_0 = \sqrt{k_i / m} = 528.787 \text{ rd/s} \] and \( \omega_{10} = \sqrt{k_i / m} = 579.176 \text{ rd/s} \) \tag{46}

\[
\begin{align*}
\omega_1 &= \sqrt{30.7531 \times 10^4 + 2.86 \times 10^4 \Omega^2 - \sqrt{30.7531 \times 10^4 + 2.86 \times 10^4 \Omega^2}} - 2.3449 \times 10^4 \\
\omega_2 &= \sqrt{30.7531 \times 10^4 + 2.86 \times 10^4 \Omega^2 + \sqrt{30.7531 \times 10^4 + 2.86 \times 10^4 \Omega^2}} - 2.3449 \times 10^4 
\end{align*}
\tag{47} \]

A: \( \Omega_{c1} = 492.917 \text{ rd/s} \); B: \( \Omega_{c2} = 639.857 \text{ rd/s} \),
C: \( \Omega_{c1} = 906.314 \text{ rd/s} \); D: \( \Omega_{c2} = 1772.77 \text{ rd/s} \).
In the free-simply supported case, 
- Rotor with one disc:
\[
\omega^4 - (3.5237 \times 10^5 + 2.4619 \times 10^{-4} \Omega^2) \omega^2 + 3.0883 \times 10^{12} = 0
\]  
(48)
and
\[
\omega_{10} = \sqrt{k_1 / m} = 1279.0459 \text{ rd/s}
\]
(49)
\[
\omega_{20} = \sqrt{k_1 / m} = 1373.962 \text{ rd/s}
\]
(50)
A: \(\Omega_{c1} = 1278.036 \text{ rd/s} \); B: \(\Omega_{c2} = 1375.2168 \text{ rd/s} \),
C: \(\Omega_{c1} = 2550.3034 \text{ rd/s} \); D: \(\Omega_{c2} = 2757.667 \text{ rd/s} \),
- Rotor with two discs:
\[
\omega^4 - (21.8110^5 + 1.1683 \times 10^{-3} \Omega^2) \omega^2 + 1.18327 \times 10^{12} = 0
\]  
(51)
\[
\omega_{10} = \sqrt{k_1 / m} = 1006.299 \text{ rd/s}
\]
(52)
\[
\omega_{20} = \sqrt{k_1 / m} = 1080.975 \text{ rd/s}
\]
(53)
A: \(\Omega_{c1} = 1002.69 \text{ rd/s} \); B: \(\Omega_{c2} = 1085.49 \text{ rd/s} \),
C: \(\Omega_{c1} = 1987.43 \text{ rd/s} \); D: \(\Omega_{c2} = 2194.46 \text{ rd/s} \).
- Rotor with three discs:
\[
\omega^4 - (21.8110^5 + 1.1683 \times 10^{-3} \Omega^2) \omega^2 + 1.18327 \times 10^{12} = 0
\]  
(54)
\[
\omega_{10} = \sqrt{k_1 / m} = 1006.299 \text{ rd/s}
\]
(55)
\[
\omega_{20} = \sqrt{k_1 / m} = 1080.975 \text{ rd/s}
\]
(56)
A: \(\Omega_{c1} = 900.89 \text{ rd/s} \); B: \(\Omega_{c2} = 1058.6 \text{ rd/s} \),
C: \(\Omega_{c1} = 1705.27 \text{ rd/s} \); D: \(\Omega_{c2} = 2311.056 \text{ rd/s} \).

5. RESPONSE TO A SYNCHRONOUS FORCE

When considering the excitation force due to the mass-unbalance, the steady state solution which is the particular solution is found by solving the inhomogeneous system of equations (17) i.e the system with second member. For this one, the solutions of the system may have the form:

\[
\begin{align*}
q_{1p} &= A_{el} \cos(\Omega t + \phi_{el}) \\
q_{2p} &= A_{e2} \cos(\Omega t + \phi_{e2})
\end{align*}
\]  
(57)
They can be transformed in a complex form as:

\[
\begin{align*}
q_{1p} &= A_{el} \exp j\Omega t \\
q_{2p} &= A_{e2} \exp j\Omega t
\end{align*}
\]  
(58)
With: \(A_{el} = A_{el} \exp j\phi_{el}\) and \(A_{e2} = A_{e2} \exp j\phi_{e2}\).

Their introduction in the inhomogeneous complex system yields:

\[
\begin{align*}
\bar{m}q_{1p} - a\Omega q_{2p} + k_1 q_{1p} &= C\Omega^2 \exp j(\Omega t - \pi / 2) \quad \text{(59)} \\
\bar{m}q_{2p} + a\Omega q_{1p} + k_2 q_{2p} &= C\Omega^2 \exp j(\Omega t)
\end{align*}
\]
or

\[
\begin{align*}
k_1 - m\Omega^2 - j\Omega^2 &= \frac{A_{el}}{C\Omega^2} \quad \text{(60)}
k_2 - m\Omega^2 - j\Omega^2 &= \frac{A_{e2}}{C\Omega^2}
\end{align*}
\]
This last system of equations represents a linear system with two unknowns \(A_{el}\) et \(A_{e2}\) that depend on a parameter \(\Omega\). The determinant method gives:

\[
A_{el} = \frac{C\Omega^2 e^{-j\pi/2} - j\Omega^2}{k_1 - m\Omega^2 - j\Omega^2} = \frac{C\Omega^2 (k_2 - m\Omega^2) e^{-j\pi/2} + jaC\Omega^4}{[k_1 - m\Omega^2](k_2 - m\Omega^2) - a^2\Omega^4}
\]
(61)
\[
A_{e2} = \frac{C\Omega^2 e^{-j\pi/2} - j\Omega^2}{k_1 - m\Omega^2 - j\Omega^2} = \frac{C\Omega^2 (k_2 - m\Omega^2) e^{-j\pi/2} + jaC\Omega^4}{[k_1 - m\Omega^2](k_2 - m\Omega^2) - a^2\Omega^4}
\]
(62)
From these relations, we obtain:

\[
A_{el} = \frac{k_2 - (m + a)\Omega^2}{(k_1 - m\Omega^2)(k_2 - m\Omega^2) - a^2\Omega^4}
\]
(63)
\[
A_{e2} = \frac{k_1 - (m + a)\Omega^2}{(k_1 - m\Omega^2)(k_2 - m\Omega^2) - a^2\Omega^4}
\]
Also \(\phi_{el} = -\pi/2\) and \(\phi_{e2} = 0\).
So we can write:

\[ q_{1p} = A_{e1} \sin \Omega t \quad \text{and} \quad q_{2p} = A_{e2} \cos \Omega t \]  

(64)

As \( A_{e1} \) et \( A_{e2} \) are different, the orbits described by the rotor are ellipses. The expressions of the critical speeds come from the denominator of the equation (62) when it is equal to zero, that is:

\[(m^2 - a^2)\Omega^4 - m(k_1 + k_2)\Omega^2 + k_1k_2 = 0 \]  

(65)

We remark that this equation is similar to equation (38) for the case where \( s = 1 \). The two critical speeds of rotation correspond to the points A and B. However the sense of precession is given by the product of the amplitudes. Indeed, if \( A_{e1}A_{e2} > 0 \) the precession is direct and if \( A_{e1}A_{e2} < 0 \) the precession is reverse.

In order to know the sign of \( A_{e1}A_{e2} \), we have to know the sign of the following function:

\[ f(\Omega^2) = k_1 - (m + a)\Omega^2 + k_2 - (m + a)\Omega^2 \]  

(66)

It is equal to zero for the two values of \( \Omega \):

\[ \Omega_1 = \sqrt{k_1/(m + a)} \quad \text{and} \quad \Omega_2 = \sqrt{k_2/(m + a)} \]

These allow having the sense of the precession for each value of the rotational speed (see figure 4).

![Figure 4](image)

**Figure 4** : Variation of the sense of the precession

For the different cases of the considered model of rotor defined by equations (11) to (16), we have:

In the simply-simply supported case,

- Rotor with one disc:

\[ A_{e1} = \frac{486,220 - 4,415.10^{-4} \Omega^2}{2185,706}\Omega^4 - 44,59.10^{10} \Omega^2 + 2,255.10^{15} \]  

\[ A_{e2} = \frac{405,297 - 4,415.10^{-4} \Omega^2}{2185,706}\Omega^4 - 44,59.10^{10} \Omega^2 + 2,255.10^{15} \]  

(67)

- Rotor with two discs:

\[ A_{e1} = \frac{486,220 - 1,371.10^{-3} \Omega^2}{110,646.10^{10}\Omega^4 - 1,059.10^{10} \Omega^2 + 2,255.10^{15} \]  

\[ A_{e2} = \frac{405,297 - 1,371.10^{-3} \Omega^2}{110,646.10^{10}\Omega^4 - 1,059.10^{10} \Omega^2 + 2,255.10^{15} \]  

(68)

- Rotor with three discs:

\[ A_{e1} = \frac{521,055 - 1,924.8.10^{-3} \Omega^2}{22,668.10^{10}\Omega^4 - 1,478.8.10^{10} \Omega^2 + 2,255.10^{15} \]  

\[ A_{e2} = \frac{434,334 - 1,924.8.10^{-3} \Omega^2}{22,668.10^{10}\Omega^4 - 1,478.8.10^{10} \Omega^2 + 2,255.10^{15} \]  

(69)

In the free-simply supported case,

- Rotor with one disc:

\[ A_{e1} = \frac{973,875 - 5,178.6.10^{-4} \Omega^2}{2703,818}\Omega^4 - 96,576.5.10^8 \Omega^2 + 8,577.10^{15} \]  

\[ A_{e2} = \frac{843,968 - 5,178.6.10^{-4} \Omega^2}{2703,818}\Omega^4 - 96,576.5.10^8 \Omega^2 + 8,577.10^{15} \]  

(70)

- Rotor with two discs:

\[ A_{e1} = \frac{973,875 - 8,619.10^{-4} \Omega^2}{7240,352}\Omega^4 - 1,581.10^{10} \Omega^2 + 8,577.10^{15} \]  

\[ A_{e2} = \frac{843,968 - 8,619.10^{-4} \Omega^2}{7240,352}\Omega^4 - 1,581.10^{10} \Omega^2 + 8,577.10^{15} \]  

(71)

- Rotor with three discs:

\[ A_{e1} = \frac{3700,8 - 4,173.8.10^{-3} \Omega^2}{9430,576}\Omega^4 - 1,822.10^{10} \Omega^2 + 8,577.10^{15} \]  

\[ A_{e2} = \frac{3207,49 - 4,173.8.10^{-3} \Omega^2}{9430,576}\Omega^4 - 1,822.10^{10} \Omega^2 + 8,577.10^{15} \]  

(72)

\( A_{e1} \) and \( A_{e2} \) are represented in absolute values on figure 5 and figure 6, where we see that the phenomenon of resonance occurs for two critical values unlike the symmetric case where it occurs for a single value. On the other hand, when \( \Omega >> 0 \) (very positive), the amplitudes \( A_{e1} \) et \( A_{e2} \) will be equal and tend to a constant value equals to: 9.833810^-m.

6. **RESPONSE TO AN ASYNCHRONOUS FORCE**

The rotor can also be excited by an asynchronous force during its operation. It is a force with a constant amplitude \( F_0 \) and speeds \( s \Omega \) different from that of the rotor. If the force is applied in \( l_3 \) we have:

\[ F_{q1} = F_0 f(l_3) \sin s\Omega t = F \sin s\Omega t \]

and \( F_{q2} = F_0 f(l_3) \cos s\Omega t = F \cos s\Omega t \)  

(73)

The equations to be resolved will be:

\[ \begin{cases} m\ddot{q}_1 - a\Omega \dot{q}_2 + k_1q_1 = F \sin s\Omega t \\ m\ddot{q}_2 + a\Omega \dot{q}_1 + k_2q_2 = F \cos s\Omega t \end{cases} \]  

(74)
As the precedent case, the solutions are of the form:

\[
A_{e_1} = \frac{[k_2 - (ms^2 + as)\Omega^2]F}{s^5 (s^2 m^2 - a^2) \Omega^3 - ms^2 (k_1 + k_2) \Omega^2 + k_1 k_2}
\]

And

\[
A_{e_2} = \frac{[k_1 - (mh^2 + as)\Omega^2]F}{s^5 (s^2 m^2 - a^2) \Omega^3 - ms^2 (k_1 + k_2) \Omega^2 + k_1 k_2}
\]

The critical speeds are given by the same equation (38) and the orbits are described by ellipses.
\begin{align}
q_{1p} &= A_{e1} \cos(s\Omega t + \phi_{e1}) \\
q_{2p} &= A_{e2} \cos(s\Omega t + \phi_{e2})
\end{align}

(76)

The calculations lead us to:

In the simply-simply supported case,

- Rotor with one disc:

- Rotor with two discs:

- Rotor with three discs:

Figure 7(a) : Response to an asynchronous force: rotor with one disc simply-simply supported case.

Figure 7(b) : Response to an asynchronous force: rotor with two discs simply-simply supported case.

Figure 7(c) : Response to an asynchronous force: rotor with three discs simply-simply supported case.

Figure 8(a) : Response to an asynchronous force: rotor with one disc free-simply supported case.

Figure 8(b) : Response to an asynchronous force: rotor with two discs free-simply supported case.

Figure 8(c) : Response to an asynchronous force: rotor with three discs free-simply supported case.
Flexural vibration of a multi-disc rotor with different boundary conditions

\[ \begin{align*}
A_{e1} &= \frac{(43,358.10^6 - 45,574 \Omega^2)F}{453,636 \Omega^4 - 26,479.10^4 \Omega^2 + 2,255.10^{15}} \\
A_{e2} &= \frac{(52,015.10^6 - 57,31\Omega^2)F}{1158,86\Omega^4 - 36,9719.10^4 \Omega^2 + 2,255.10^{15}} \\
A_{e3} &= \frac{(43,358.10^6 - 57,31\Omega^2)F}{1158,86\Omega^4 - 36,9719.10^4 \Omega^2 + 2,255.10^{15}} \\
\end{align*} \]

- Rotor with three discs:

\[ \begin{align*}
A_{e1} &= \frac{(99,4867.10^6 - 13,588\Omega^2)F}{173,4136\Omega^4 - 24,466.10^4 \Omega^2 + 8,577.10^{15}} \\
A_{e2} &= \frac{(86,216.10^6 - 13,588\Omega^2)F}{173,4136\Omega^4 - 24,466.10^4 \Omega^2 + 8,577.10^{15}} \\
A_{e3} &= \frac{(99,4867.10^6 - 22,74\Omega^2)F}{450,934\Omega^4 - 39,5268.10^4 \Omega^2 + 8,577.10^{15}} \\
A_{e4} &= \frac{(86,216.10^6 - 22,74\Omega^2)F}{450,934\Omega^4 - 39,5268.10^4 \Omega^2 + 8,577.10^{15}} \\
\end{align*} \]

- Rotor with two discs:

\[ \begin{align*}
A_{e1} &= \frac{(99,4867.10^6 - 31,569\Omega^2)F}{552,261\Omega^4 - 45,555.10^4 \Omega^2 + 8,577.10^{15}} \\
A_{e2} &= \frac{(86,216.10^6 - 31,569\Omega^2)F}{552,261\Omega^4 - 45,555.10^4 \Omega^2 + 8,577.10^{15}} \\
\end{align*} \]

- Rotor with one disc:

The amplitudes \(A_{e1}\) and \(A_{e2}\) for \(F=1\), are represented in figures (7, 8) where the critical speeds are the speeds that make the magnitude infinite and the denominator of a zero value.

**CONCLUSION**

In this work we have investigated the effect of the change in the boundary conditions on the vibration behavior of flexible rotor. The supported-free case is examined and compared with the supported-supported case of Lalanne and Ferrari.

On the one hand the results obtained by the Campbell diagram showed a net difference of critical speeds for these two cases.

On the other hand forces synchronous and asynchronous responses have given separate curves where you notice for the supported-free case of frequencies of overtones than those of the cases supported-supported.

It gives the possibility to work with greater speeds. Also it may be noted that playing on the boundaries conditions to allow avoiding the critical speeds for a given rotational speed and without changing the rotor structure.

**REFERENCES**