ESTIMATION FOR BOUNDED SOLUTIONS OF INTEGRAL INEQUALITIES SOME NEW NON-LINEAR RETARDER INTEGRO-DIFFERENTIAL INEQUALITIES.

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Abstract

In this paper, we establish some new non-linear retarded integro-differential inequalities in tow and n independent variables.

Keywords: boundary Value Problems ; Retard integro-differential Equations ; Partial integro-differential equations ; integral inequalities.

INTRODUCTION

The study of integral inequalities involving functions of one or more independent variables is an important tool in the study of existence, uniqueness, bounds, stability, invariant manifolds and other qualitative properties of solutions of differential equations and integral equations (see : [1-6, 8,12]).

The study of integro-differential inequalities for functions of two or n variables is very significant and plays a role in the study of the existence and uniqueness of the solutions of Wendroff type integro-differential inequalities and equations as well as the boundedness of the solutions of the initial value problem of non-linear retarded hyperbolic partial integro-differential equations for functions of two or n variables [9-11].


Lemma 1. (See Theorem 1 [7]) Let \( \Phi(x,y) \) and \( c(x,y) \) be non-negative continuous functions defined for \( x \geq 0, y \geq 0 \), and \( \Phi(0,0) = \Phi(x,0) = 0 \) for which the inequality

\[
\Phi_{xy}(x,y) \leq a(x) + b(y)
\]

\[
+ \int_0^x \int_0^y c(s,t) \left( \Phi(s,t) + \Phi_{xy}(s,t) \right) dsdt,
\]

holds for \( x \geq 0, y \geq 0 \), where \( a(x), b(y) \geq 0 \); \( a'(x), b'(y) \geq 0 \) are continuous functions defined for \( x \geq 0, y \geq 0 \). Then

\[
\Phi_{xy}(x,y) \leq a(x) + b(y) + \int_0^x \int_0^y c(s,t) \left[ \frac{a(0)+b(t)}{a(s)+b(0)} \right] \exp \left( \int_0^s \int_0^t [1 + c(\tau,\sigma)] d\tau d\sigma \right) dsdt.
\]

Our main aim here, motivated by the works of Pachpatte [7] , Zhang, H. and Meng [12], is to establish some new non-linear retarded integro-differential inequalities for functions with tow and n independent variables which are useful in the analysis of certain classes of partial differential equations and integro-differential inequalities. Some applications of our results are also given.

Throughout this paper, we denote \( \mathbb{R}_+^n = [0, \infty[ \times \cdots \times [0, \infty[ \), which is a subset of \( \mathbb{R}_+^n \), \( n \geq 1 \). All the functions which appear in the inequalities are assumed to be real valued of \( n -\)variables \( n \geq 1 \) which are non-negative and continuous. All integrals are assumed to exist on their domains of definitions.

We note \( \mathcal{D} = \mathcal{D}_1 \mathcal{D}_2 \cdots \mathcal{D}_n \), where \( \mathcal{D}_i \), for \( i = 1, 2, \cdots, n \).

II. MAIN RESULTS

In this section, we present some results of non-linear retarded integro-differential inequalities in two independent variables.

Theorem 2. Let \( u(x,y), c(x,y) \) and \( a(x,y), D_i u(x,y) \) and \( D_{\alpha} u(x,y) \) be non-negative continuous functions for all \( i = 1, 2 \) defined for \( x, y \in \mathbb{R}_+ \) and \( a, \beta \in C^1(\mathbb{R}_+, \mathbb{R}_+) \) be non-decreasing functions in each variable, with \( a(x) \geq x \) on \( \mathbb{R}_+ \), and \( \beta(y) \geq y \) on \( \mathbb{R}_+ \). Let \( c(x,y) \) be non-decreasing in each variable \( x, y \in \mathbb{R}_+ \), and

\[
\lim_{x \to \infty} u(x,y) = \lim_{x \to \infty} u(x,y) = 0.
\]

If

\[
Du(x,y) \leq c(x,y)
\]

\[
+ \int_0^x \int_0^y a(s,t)[u(s,t)] dsdt
\]

\[
+ Du(s,t) \right] dsdt,
\]

for all \( x, y \in \mathbb{R}_+ \), then

\[
Du(x,y) \leq c(x,y)
\]

\[
\leq \int_0^x \int_0^y a(s,t) \exp \left( \int_0^\infty \int_0^\infty [a(\tau,\sigma)] d\tau d\sigma \right) dt.
\]

For all \( x, y \in \mathbb{R}_+ \).

Proof: Fix any \( X, Y \in \mathbb{R}_+ \). Then, for \( x \leq X \) and \( y \leq Y \), we have...
Define a function \( z(x, y) \) by

\[
z(x, y) = c(X, Y) + \int_{\alpha(x)}^{\infty} \int_{\beta(y)}^{\infty} a(s, t)[u(s, t) + Du(s, t)]dsdt,
\]

(2.3)

Then

\[
\lim_{x \to \infty} z(x, y) = \lim_{x \to \infty} z(x, y) = c(X, Y),
\]

(2.4)

By differentiating (2.3)

\[
Dz(x, y) \leq a\{a(x, \beta(y))\} \{u(a(x, \beta(y))) + Du(a(x, \beta(y)))\} \alpha'(x) \beta'(y).
\]

(2.5)

By integrating both sides of (2.4)

\[
Du(x, y) \leq \int_{\alpha(x)}^{\infty} \int_{\beta(y)}^{\infty} z(s, t) dsdt,
\]

(2.6)

Now, using (2.4) and (2.6) in (2.5), we get

\[
Du(x, y) \leq a\{a(x, \beta(y))\} \{z(x, y) + \int_{\alpha(x)}^{\infty} \int_{\beta(y)}^{\infty} z(s, t) dsdt\} \alpha'(x) \beta'(y).
\]

(2.7)

If we put

\[
v(x, y) = z(x, y) + \int_{\alpha(x)}^{\infty} \int_{\beta(y)}^{\infty} z(s, t) dsdt,
\]

(2.8)

then

\[
\lim_{x \to \infty} z(x, y) = \lim_{y \to \infty} z(x, y) = c(X, Y),
\]

and

\[
Du(x, y) \leq Dz(x, y) + z(x, y) \alpha'(x) \beta'(y).
\]

(2.9)

It is easy to estimate \( v(x, y) \) by

\[
v(x, y) \leq c(X, Y) \int_{\alpha(x)}^{\infty} \int_{\beta(y)}^{\infty} [1 + a(s, t)] dsdt.
\]

(2.10)

By substituting (2.9) in (2.7) and integrating both sides, we get

\[
z(x, y) \leq c(X, Y) \int_{\alpha(x)}^{\infty} \int_{\beta(y)}^{\infty} a(s, t) exp \left[ \int_{\infty}^{s} a(r, \sigma) drds \right] dsdt.
\]

(2.11)

Since \( X \) and \( Y \) are arbitrarily chosen and by substituting the value of \( z(x, y) \) in (2.4), we obtain the inequality (2.2).

**Remark 1** If \( x = 0, c(x) = \alpha, \beta(y) = \gamma, \) and \( c(x, y) = c_1(x) + c_2(y) \) in Theorem 2 we obtain Theorem 1 in [7].

**Corollary 3.** Let \( u(x, y), c(x, y) \) and \( a(x, y), D_t u(x, y) \) and \( Du(x, y) \) be non-negative continuous functions for all \( i = 1, 2, 3 \) defined for \( x, y \in \mathbb{R}_+ \) and \( \alpha, \beta \in C(\mathbb{R}_+, \mathbb{R}_+) \) be non-decreasing functions in each variable, with \( \alpha(x) \geq x \) and \( \beta(y) \geq y \) on \( \mathbb{R}_+ \). Let \( c(x, y) \) be non-decreasing in each variable \( x, y \in \mathbb{R}_+ \), and

\[
\lim_{x \to \infty} u(x, y) = \lim_{y \to \infty} u(x, y) = 0,
\]

then

\[
Du(x, y) \leq \int_{\alpha(x)}^{\infty} \int_{\beta(y)}^{\infty} a(s, t) dsdt,
\]

(2.12)

Define a function \( z(x, y) \) by

\[
z(x, y) = c(X, Y) \int_{\alpha(x)}^{\infty} \int_{\beta(y)}^{\infty} a(s, t) exp \left[ \int_{\infty}^{s} a(r, \sigma) drds \right] dsdt.
\]

(2.13)

By differentiating (2.12) and using (2.13), we have

\[
Du(x, y) \leq z(x, y) \int_{\alpha(x)}^{\infty} \int_{\beta(y)}^{\infty} a(s, t) exp \left[ \int_{\infty}^{s} a(r, \sigma) drds \right] dsdt.
\]

Therefore

\[
z(x, y) \leq c(X, Y) \int_{\alpha(x)}^{\infty} \int_{\beta(y)}^{\infty} [M + Ma(a(x, \beta(y))) + a(a(x, \beta(y)))] a'(x) \beta'(y).
\]

(2.14)
Since $X$ and $Y$ are arbitrary and by substituting the value of $z(x, y)$ in (2.13), we obtain the inequality (2.11).

**Remark 2.** If we put $\infty = 0$, $\alpha(x) = x$, $\beta(y) = y$, and $c(x, y) = c_1(x) + c_2(y)$ in corollary 3 we obtain theorem 2 in [8].

**Corollary 4.** Let $u(x, y), c(x, y)$ and $a(x, y), D_iu(x, y)$ and $Du(x, y)$ be non-negative continuous functions for all $i = 1, 2$ defined for $x, y \in \mathbb{R}_+$ and $\alpha, \beta \in C^1(\mathbb{R}_+, \mathbb{R}_+)$ be non-decreasing functions in each variable, with $\alpha(x) \geq x$ on $\mathbb{R}_+$, and $\beta(y) \geq y$ on $\mathbb{R}_+$. Let $c(x, y)$ be non-decreasing in each variable $x, y \in \mathbb{R}_+$, and

$$
\lim_{x \to \infty} u(x, y) = \lim_{x \to \infty} u(x, y) = 0,
$$

for all $x, y \in \mathbb{R}_+$, where $M > 0$ is constant, then

$$
Du(x, y) \leq c(x, y) \exp \left[ \int_{\alpha(x)}^{\infty} \int_{\beta(y)}^{\infty} M[a(s, t) + 1]dsdt \right]
$$

for all $x, y \in \mathbb{R}_+$. Proof: The proof of this Corollary follows the same arguments as in Corollary 3.

**Remark 3.** If we put $\infty = 0$, $\alpha(x) = x$, $\beta(y) = y$, and $c(x, y) = c_1(x) + c_2(y)$ in Corollary 4 we obtain the result in [12].

**Theorem 5.** Let $u(x, y), c(x, y)$ and $a(x, y), b(x, y)$ be non-negative continuous functions defined for $x, y \in \mathbb{R}_+$ and $\alpha, \beta \in C^1(\mathbb{R}_+, \mathbb{R}_+)$ be non-decreasing functions in each variable, with $\alpha(x) \geq x$ on $\mathbb{R}_+$, and $\beta(y) \geq y$ on $\mathbb{R}_+$. Let $c(x, y)$ be non-decreasing in each variable $x, y \in \mathbb{R}_+$. If

$$
u(x, y) \leq c(x, y) + \int_{\alpha(x)}^{\infty} \int_{\beta(y)}^{\infty} a(s, t)u(s, t)dsdt \tag{2.14}$$

and

$$
\int_{\alpha(x)}^{\infty} \int_{\beta(y)}^{\infty} a(s, t) \left[ \int_{s}^{\infty} b(\tau, \sigma)u(\tau, \sigma)d\tau \right] d\sigma
$$

for all $x, y \in \mathbb{R}_+$, then

$$
u(x, y) \leq c(x, y) \exp \left[ \int_{\alpha(x)}^{\infty} \int_{\beta(y)}^{\infty} a(s, t)dsdt \right.

+ \int_{\alpha(x)}^{\infty} \int_{\beta(y)}^{\infty} a(s, t) \left[ \int_{s}^{\infty} b(\tau, \sigma)d\tau \right] d\sigma \left. \right] d\sigma, \tag{2.15}
$$

for all $x, y \in \mathbb{R}_+$.

Proof: Since $c(x, y)$ is non-negative and non-decreasing, from (2.14) we have

$$
\frac{\nu(x, y)}{c(x, y)} \leq 1 + \int_{\alpha(x)}^{\infty} \int_{\beta(y)}^{\infty} \frac{a(s, t)}{c(s, t)} \frac{u(s, t)}{c(s, t)} dsdt + \int_{\alpha(x)}^{\infty} \int_{\beta(y)}^{\infty} a(s, t) \left[ \int_{s}^{\infty} b(\tau, \sigma) \frac{u(\tau, \sigma)}{c(\tau, \sigma)} d\tau \right] d\sigma d\sigma,
$$

Define a function $z(x, y)$ by the right side of the last inequality. Then $z(x, y) \geq 0$,

$$
\lim_{x \to \infty} z(x, y) = \lim_{x \to \infty} z(x, y) = 1, \frac{u(x, y)}{c(x, y)} \leq z(x, y),
$$

and

$$
Dz(x, y) \leq z(x, y) \left[ a(x, y) + \int_{\alpha(x)}^{\infty} \int_{\beta(y)}^{\infty} b(s, t)dsdt \right] \alpha'(x)\beta'(y).
$$

i.e

$$
\frac{Dz(x, y)}{z^2(x, y)} \leq \left[ a(x, y) + \int_{\alpha(x)}^{\infty} \int_{\beta(y)}^{\infty} b(s, t)dsdt \right] \alpha'(x)\beta'(y).
$$

Thus

$$
D_2 \left[ \frac{D_2z(x, y)}{z(x, y)} \right] \leq \left[ a(x, y) + \int_{\alpha(x)}^{\infty} \int_{\beta(y)}^{\infty} b(s, t)dsdt \right] \alpha'(x)\beta'(y).
$$

By keeping $y$ fixed, setting $x = s$, and integrating from $a(x)$ to $\infty$ in (2.16), and again by keeping $x$ fixed, setting $y = t$, and integrating from $\beta(y)$ to $\infty$ in the resulting inequality, we have

$$
z(x, y) \leq c(x, y) \exp \int_{\alpha(x)}^{\infty} \int_{\beta(y)}^{\infty} a(s, t)dsdt + \int_{\alpha(x)}^{\infty} \int_{\beta(y)}^{\infty} a(s, t) \left[ \int_{s}^{\infty} b(\tau, \sigma)d\tau \right] d\sigma d\sigma.
$$

Finally, since

$$
\frac{u(x, y)}{c(x, y)} \leq z(x, y)
$$

We obtain the inequality (2.15).

**Remark 4.**

1. If we put $\infty = 0$, $\alpha(x) = x$, $\beta(y) = y$, and $c(x, y) = c_1(x) + c_2(y)$ in theorem 5 we obtain theorem 3 [7].

2. In the particular case when $b(x, y) = 0$, then the bound obtained in [8] reduces to

$$
u(x, y) \leq c(x, y) \exp \int_{\alpha(x)}^{\infty} \int_{\beta(y)}^{\infty} a(s, t)dsdt. \tag{2.17}
$$

**Theorem 6.** Let $u(x, y), c(x, y)$ and $a(x, y), b(x, y), f(x, y), D_iu(x, y)$ and $Du(x, y)$ be non-negative continuous functions for all $i = 1, 2$ defined for $x, y \in \mathbb{R}_+$ and $\alpha, \beta \in C^1(\mathbb{R}_+, \mathbb{R}_+)$ be non-decreasing functions in each variable, with $\alpha(x) \geq x$ on $\mathbb{R}_+$, and $\beta(y) \geq y$ on $\mathbb{R}_+$.

And
\[
\lim _{x \to \infty } u(x, y) = \lim _{x \to \infty } u(x, y) = 0
\]

Let \( K(u(x, y)) \) be a real-valued, positive, continuous, strictly non-decreasing, sub-additive, and sub-multiplicative function for \( u(x, y) \geq 0 \), and \( H(u(x, y)) \) be a real-valued, positive, continuous and non-decreasing function defined for \( x, y \in \mathbb{R}_+ \). Assume that \( c(x, y) \) and \( f(x, y) \) are non-decreasing in each of the variables \( x, y \in \mathbb{R}_+ \). If
\[
Du(x, y) \leq c(x, y) + f(x, y)H \left( \int _{a(x)}^{\infty } \int _{b(y)}^{\infty } a(s, t)K(u(s, t))dsdt \right) + \int _{a(x)}^{\infty } \int _{b(y)}^{\infty } b(s, t)Du_u(s, t)dsdt ,
\] (2.18)

for all \( x, y \in \mathbb{R}_+ \), then
\[
u(x, y) \leq c(x, y) + f(x, y)H \left( 1 + \int _{a(x)}^{\infty } \int _{b(y)}^{\infty } a(s, t)K(u(s, t))dsdt \right) p(x, y)
\]

for all \( x, y \in \mathbb{R}_+ \), where
\[
p(x, y) = \int _{a(x)}^{\infty } \int _{b(y)}^{\infty } \exp \left[ \int _{t}^{\infty } b(t, \sigma )d\sigma \right] dsdt .
\] (2.20)

\[
\xi = \int _{0}^{\infty } a(s, t)K(c(s, t)p(s, t))dsdt .
\] (2.21)

\[
G(r) = \int _{r}^{\infty } \frac{ds}{K(H(s))} .
\] (2.22)

Where \( G^{-1} \) is the inverse function of \( G \), and
\[
G(\xi) + \int _{a(x)}^{\infty } \int _{b(y)}^{\infty } a(s, t)K(f(s, t)p(s, t))dsdt \in \text{dom}(G^{-1})
\]

for all \( x, y \in \mathbb{R}_+ \).

**Proof**: Define a function \( z(x, y) \) by
\[
z(x, y) = c(x, y) + f(x, y)H \left( \int _{a(x)}^{\infty } \int _{b(y)}^{\infty } a(s, t)K(u(s, t))dsdt \right) .
\] (2.23)

then from (2.18), we have
\[
Du(x, y) \leq z(x, y) + \int _{a(x)}^{\infty } \int _{b(y)}^{\infty } b(s, t)Du(s, t)dsdt .
\] (2.24)

Clearly, \( z(x, y) \) is a positive, continuous, and decreasing function in each of the variables \( x, y \in \mathbb{R}_+ \). Using (2.17) from Theorem 5 in (2.24), we get
\[
Du(x, y) \leq z(x, y)\exp \left( \int _{a(x)}^{\infty } \int _{b(y)}^{\infty } b(s, t)dsdt \right) .
\] (2.25)

By integration, first, with respect to \( x \) from \( x \) to \( \infty \), and then with respect to \( y \) from \( y \) to \( \infty \) in the last inequality, we obtain
\[
\lim _{x \to \infty } \int _{a(x)}^{\infty } \int _{b(y)}^{\infty } b(s, t)dsdt = 0 .
\]

where \( p(x, y) \) is defined in (2.20). From (2.23) we have
\[
\int _{a(x)}^{\infty } \int _{b(y)}^{\infty } a(s, t)K(u(s, t))dsdt + \int _{a(x)}^{\infty } \int _{b(y)}^{\infty } b(s, t)Du(s, t)dsdt
\]

Now, using (2.27) in (2.26) we get
\[
\lim _{x \to \infty } \int _{a(x)}^{\infty } \int _{b(y)}^{\infty } a(s, t)K(u(s, t))dsdt .
\] (2.28)

From (2.28) and (2.29) and since \( K \) is a sub-additive and sub-multiplicative function, we obtain
\[
\int _{a(x)}^{\infty } \int _{b(y)}^{\infty } a(s, t)K(f(s, t)p(s, t))dsdt \leq \int _{a(x)}^{\infty } \int _{b(y)}^{\infty } a(s, t)K(c(s, t)p(s, t))dsdt + \int _{a(x)}^{\infty } \int _{b(y)}^{\infty } a(s, t)K(f(s, t)H(v(s, t)))dsdt .
\]

Therefore
\[
\int _{a(x)}^{\infty } \int _{b(y)}^{\infty } a(s, t)K(c(s, t)p(s, t))dsdt + \int _{a(x)}^{\infty } \int _{b(y)}^{\infty } a(s, t)K(f(s, t)p(s, t))K \left( H(v(s, t)) \right)dsdt .
\]

Define a function \( \Phi(x, y) \) by
\[
\Phi(x, y) = \int _{a(x)}^{\infty } \int _{b(y)}^{\infty } a(s, t)K(c(s, t)p(s, t))dsdt + \int _{a(x)}^{\infty } \int _{b(y)}^{\infty } a(s, t)K(f(s, t)p(s, t))K \left( H(v(s, t)) \right)dsdt .
\] (2.30)

Then
\[
\lim _{x \to \infty } \Phi(x, y) = \lim _{y \to \infty } \Phi(x, y)
\] (2.31)

and
\[
\Phi(x, y) \leq \Phi(x, y) .
\] (2.32)

Clearly, \( \Phi(x, y) \) is a positive and decreasing function in \( y \), then
\[
D_y \Phi(x, y) = - \int _{b(y)}^{\infty } a(x, t)K(f(a(x, t)p(a(x, t)))K \left( H(v(a(x, t))) \right)dsdt a'(x)
\]

\[
\geq -K \left( H(\Phi(x, y)) \right) \int _{b(y)}^{\infty } a(x, t)K(f(a(x, t)p(a(x, t)))dsdt a'(x).
\]
i.e
\[
\frac{D_t \Phi(x,y)}{K \left( H(\Phi(x,y)) \right)} \geq - \int_{\beta(y)}^{\infty} a(\alpha(x),t) K(f(\alpha(x),t)p(\alpha(x),t))dsdt \tag{2.23}
\]

From (2.22) we have
\[
D_t G(\Phi(x,y)) = - D_t \Phi(x,y) = \frac{\Phi(x,y)}{K(\Phi(x,y))} \tag{2.24}
\]
\[
\geq - \int_{\beta(y)}^{\infty} a(\alpha(x),t) K(f(\alpha(x),t)p(\alpha(x),t))dsdt \tag{2.25}
\]
\[
\text{Now, by setting } x = s \text{ and integrating from } x \to \infty \text{ in (2.24), and using (2.31) we get}
\]
\[
\Phi(x,y) \leq \int_{a(x)}^{\infty} \int_{\beta(y)}^{\infty} a(s,t) K(f(s,t)p(s,t))dsdt \tag{2.26}
\]
Finally, by substituting (2.27), (2.32) and (2.25), (2.26) we obtain the inequality (2.19).

Remark 5.
1. From the inequalities (2.29), (2.32) and (2.35) in the proof of Theorem 6 we can find this inequality
\[
u(x,y) \leq c(x,y) + f(x,y) \int_{\beta(y)}^{\infty} a(s,t) K(f(s,t)p(s,t))dsdt \tag{2.31}
\]
2. If we put \( \int_{a(x)}^{\infty} \int_{\beta(y)}^{\infty} a(s,t) K(f(s,t)p(s,t))dsdt \tag{2.32} \) for all \( x, y \in \mathbb{R}_+ \), then

\[
\text{for all } x, y \in \mathbb{R}_+, \text{ then}
\]
\[
u(x,y) \leq c(x,y) + H \left( T^{-1}[T(\xi) + \int_{a(x)}^{\infty} \int_{\beta(y)}^{\infty} a(s,t) K(p(s,t))dsdt] \right) p(x,y), \tag{2.37}
\]
\[
\text{for all } x, y \in \mathbb{R}_+, \text{ where } p(x,y) \text{ and } \xi \text{ are defined in theorem 6.}
\]
\[
T(r) = \int_{r}^{\infty} ds, \tag{2.38}
\]
\[
\text{Where } T^{-1} \text{ is the inverse function of } G, \text{ and}
\]
\[
T(\xi) + \int_{a(x)}^{\infty} \int_{\beta(y)}^{\infty} a(s,t) T(p(s,t))dsdt \in \text{dom}(T^{-1}) \tag{2.39}
\]

Remark 6.
1. If we put \( f(x,y) = 1, H(x) = x \) in theorem 6 then we obtain the result in Corollary 7.
2. If we put \( \int_{a(x)}^{\infty} \int_{\beta(y)}^{\infty} a(s,t) K(f(s,t)p(s,t))dsdt \tag{2.32} \) for all \( x, y \in \mathbb{R}_+ \), and \( \alpha, \beta \in C^1(\mathbb{R}_+, \mathbb{R}_+) \) be non-decreasing functions in each variable, with \( \alpha(x) \geq x \) on \( \mathbb{R}_+ \), and \( \beta(y) \geq y \) on \( \mathbb{R}_+ \).

Corollary 7. Let \( u(x,y), c(x,y) \text{ and } a(x,y), b(x,y), D_t u(x,y) \text{ and } D_u(x,y) \) be non-negative continuous functions for all \( i = 1 \text{ defined for } x, y \in \mathbb{R}_+ \) and \( \alpha, \beta \in C^1(\mathbb{R}_+, \mathbb{R}_+) \) be non-decreasing functions in each variable, with \( \alpha(x) \geq x \) on \( \mathbb{R}_+ \), and \( \beta(y) \geq y \) on \( \mathbb{R}_+ \).

And
\[
\lim_{x \to \infty} u(x,y) = \lim_{y \to \infty} u(x,y) = 0. \tag{2.40}
\]

Let \( K(u(x,y)) \) be a real-valued, positive, continuous, strictly non-decreasing, sub-additive, and sub-multiplicative function for \( u(x,y) \geq 0 \). Assume that \( c(x,y) \) is non-decreasing in each of the variables \( x, y \in \mathbb{R}_+ \).

If \( D_t u(x,y) \)
\[
\leq c(x,y) + \int_{a(x)}^{\infty} \int_{\beta(y)}^{\infty} a(s,t) K(u(s,t))dsdt \tag{2.36}
\]
\[
+ \int_{a(x)}^{\infty} b(s,t) D_t u(s,t)dsdt,
\]

for all \( x, y \in \mathbb{R}_+ \), then

\[
u(x,y) \leq c(x,y) + H \left( T^{-1}[T(\xi) + \int_{a(x)}^{\infty} \int_{\beta(y)}^{\infty} a(s,t) K(p(s,t))dsdt] \right) p(x,y), \tag{2.37}
\]

for all \( x, y \in \mathbb{R}_+ \), where \( p(x,y) \) and \( \xi \) are defined in theorem 6.
Proof: By setting $K(x) = x$ and $c(x, y) = M$ in Corollary 7, we obtain the results of this Corollary.

Corollary 9. Let $u(x, y), a(x, y), b(x, y), D_i u(x, y)$ and $Du(x, y)$ be non-negative continuous functions for all $i = 1, 2$ defined for $x, y \in \mathbb{R}_+$ and $\alpha, \beta \in C^1(\mathbb{R}_+, \mathbb{R}_+)$ be non-decreasing functions in each variable, with $\alpha(x) \geq x$ on $\mathbb{R}_+$, and $\beta(y) \geq y$ on $\mathbb{R}_+$.

And

$$\lim_{x \to +\infty} u(x, y) = \lim_{x \to +\infty} u(x, y) = 0$$

Let $K(u(x, y))$ be a real-valued, positive, continuous, strictly non-decreasing, sub-additive, and sub-multiplicative function for $u(x, y) \geq 0$. If

$$Du(x, y) \leq c_1(x) + c_2(y) + \int_{a(x)}^{\beta(y)} \int_{a(x)}^{\beta(y)} a(s, t)K(u(s, t))dsdt + \int_{a(x)}^{\beta(y)} b(s, t)Du(s, t)dsdt,$$

For all $x, y \in \mathbb{R}_+$, where $c_1(x), c_2(y) > 0$, and $\alpha'(x), \beta'(y) > 0$ are continuous functions defined for $x \geq 0, y \geq 0$ then

$$Du(x, y) \leq c_1(x) + c_2(y) + \left( T^{-1} \left( T(\xi) + \int_{a(x)}^{\beta(y)} a(s, t)K(p(s, t))dsdt \right) \right) \exp \int_{a(x)}^{\beta(y)} b(s, t)dsdt,$$

For all $x, y \in \mathbb{R}_+$, where

$$\xi = \int_{0}^{\infty} a(s, t)K \left( c_1(s) + c_2(t) \right)p(s, t)dsdt$$

And $p(x, y)$ and $T$ are defined in corollary 7.

Proof: By setting $c(x, y) = c_1(x) + c_2(y)$ in Corollary 7 and using the same arguments in theorem 6, we obtain the results of this Corollary.

III. Retarded Non-Linear Integro-Differential Inequalities in n Independent Variables

In this section, we present some results of non-linear retarded integro-differential inequalities in $n$ independent variables.

In what follows, for

$$x = (x_1, x_2, ..., x_n), t = (t_1, t_2, ..., t_n), \Theta = (\infty, \infty, ..., \infty),$$

We denote :
By integrating (3.3) with respect to \( x_2 \) from \( x_2 = 0 \) to \( \alpha(x_2) \), we have
\[
\begin{align*}
&- \left( \frac{D_1 \ldots D_{n-1} z(x)}{z(x)} \right) \leq \int_{\alpha(x_2)}^{\infty} a(x_1, \ldots, x_{n-1}, t_n) d t_n a_1'(x_1) \ldots a_{n-1}'(x_{n-1}); \text{if } n \text{ is even} \\
&- \left( \frac{D_1 \ldots D_{n-1} z(x)}{z(x)} \right) \geq - \int_{\alpha(x_2)}^{\infty} a(x_1, \ldots, x_{n-1}, t_n) d t_n a_1'(x_1) \ldots a_{n-1}'(x_{n-1}); \text{if } n \text{ is odd}
\end{align*}
\]
thus
\[
\begin{align*}
&\frac{D_1 \ldots D_{n-1} z(x)}{z(x)} \leq D_{n-1} \left( \frac{D_1 \ldots D_{n-2} z(x)}{z(x)} \right); \text{if } n \text{ is even} \\
&D_1 \ldots D_{n-1} z(x) \leq D_{n-1} \left( \frac{D_1 \ldots D_{n-2} z(x)}{z(x)} \right); \text{if } n \text{ is odd}
\end{align*}
\]

By integrating the last inequality with respect to \( x_1 \) from \( x_1 = 0 \) to \( \alpha(x_1) \), we have
\[
\begin{align*}
&\frac{D_1 \ldots D_{n-2} z(x)}{z(x)} \leq \int_{\alpha(x_1)}^{\infty} \int_{\alpha(x_2)}^{\infty} a(x_1, \ldots, x_{n-2}, t_n, t_{n-1}) d t_n d t_{n-1} a_1'(x_1) \ldots a_{n-2}'(x_{n-2}); \text{if } n \text{ is even} \\
&D_1 \ldots D_{n-2} z(x) \geq - \int_{\alpha(x_1)}^{\infty} \int_{\alpha(x_2)}^{\infty} a(x_1, \ldots, x_{n-2}, t_n, t_{n-1}) d t_n d t_{n-1} a_1'(x_1) \ldots a_{n-2}'(x_{n-2}); \text{if } n \text{ is odd}
\end{align*}
\]

By continuing this process, we get
\[
\begin{align*}
&\frac{D_{n-1} z(x)}{z(x)} \leq \int_{\alpha(x_1)}^{\infty} \ldots \int_{\alpha(x_2)}^{\infty} a(x_1, t_2, \ldots, t_{n-1}, t_n) d t_n \ldots d t_2 a_1'(x_1); \text{if } n \text{ is even} \\
&D_1 z(x) \geq - \int_{\alpha(x_1)}^{\infty} \ldots \int_{\alpha(x_2)}^{\infty} a(x_1, t_2, \ldots, t_{n-1}, t_n) d t_n \ldots d t_2 a_1'(x_1); \text{if } n \text{ is odd}
\end{align*}
\]

By integrating (3.4) with respect to \( x_1 \) from \( x_1 = 0 \) to \( \alpha(x_1) \), we have
\[
z(x) \leq \exp \int_{\alpha(x)}^{\infty} a(t) d t.
\]
Finally, since \( \frac{u(x)}{c(x)} \leq z(x) \) we obtain the inequality (3.2).

Remark 7. In the particular case when \( n = 2, x \in \mathbb{R}^2_+ \), \( \varphi(x, 0, 0) = (0,0) \), \( a_1(x_1) = x_1, a_2(x_2) = x_2 \) and \( c(x) = c_1(x_1) + c_2(x_2) \) then theorem 10 reduces to lemma 1 in [12].

Theorem 11. Let \( u(x), c(x), a(x), D_1 u(x) \) and \( Du(x) \) be non-negative continuous functions for all \( i = 1,2, \ldots, n \) defined for \( x \in \mathbb{R}^n_+ \),
\[
\lim_{x_i \to \infty} u(x_1, x_2, \ldots, x_n) = 0, \forall i = 1,2, \ldots, n.
\]
Let \( \bar{a} \in C^1(\mathbb{R}^n_+, \mathbb{R}^n_+) \) be non-decreasing functions in each variable, with \( \bar{a}(x) \geq x \) on \( \mathbb{R}^n_+ \). Assume that \( c(x) \) is non-decreasing in each variable \( x \in \mathbb{R}^n_+ \). If
\[
Du(x) \leq c(x) + \int_{\bar{a}(x)}^{\infty} a(t) [u(t) + Du(t)] d t,
\]
for \( x \in \mathbb{R}^n_+ \), then
\[ u(x) \leq c(x) \left[ 1 + \int_{\tilde{a}(x)}^{\infty} a(t) \exp \int_{t}^{\infty} (1 + a(t)) dt \right] \]

(3.6)

\[ \text{Proof: Fixe any } X \in \mathbb{R}^n. \text{ Then, for } x \leq X \text{ and from (3.5), we have} \]

\[ D u(x) \leq c(X) + \int_{\tilde{a}(x)}^{\infty} a(t)[u(t) + D u(t)] dt, \]

Define a function \( z(x) \) by

\[ z(x) = c(X) + \int_{\tilde{a}(x)}^{\infty} a(t)[u(t) + D u(t)] dt, \]

Then

\[ \lim_{x_i \to \infty} z(x_1, ..., x_n) = c(X), i = 1, 2, ..., n, \]

\[ D u(x) \leq z(x), \]  

(3.8)

By differentiating (3.8)

\[ \left\{ \begin{array}{ll}
D z(x) \leq a(x)[u(x) + D u(x)] \tilde{a}'(x); & \text{if } n \text{ is even} \\
D z(x) \geq -a(x)[u(x) + D u(x)] \tilde{a}'(x); & \text{if } n \text{ is odd}
\end{array} \right. \]

(3.9)

By integrating both sides of (3.8)

\[ u(x) \leq \int_{\tilde{a}(x)}^{\infty} z(t) dt, \]

(3.10)

Now, using (3.8) and (3.10) in (3.9) we get

By substituting (3.13) in (3.11) we get

\[ \left\{ \begin{array}{ll}
D z(x) \leq a(x)c(X) \exp \left[ \int_{\tilde{a}(x)}^{\infty} (1 + a(t)) dt \right] \tilde{a}'(x); & \text{if } n \text{ is even} \\
D z(x) \geq -a(x)c(X) \exp \left[ \int_{\tilde{a}(x)}^{\infty} (1 + a(t)) dt \right] \tilde{a}'(x); & \text{if } n \text{ is odd}
\end{array} \right. \]

(3.14)

\[ \lim_{x_n \to \infty} D_{1} ... D_{n-1} z(x_1, ..., x_n, x_n) = 0. \]

By integrating (3.14) to \( x_n \) from \( x_n \) to \( \infty \), we have

\[ v(x) = z(x) + \int_{\tilde{a}(x)}^{\infty} z(t) dt, \]

(3.12)

Then

\[ \lim_{x_i \to \infty} v(x_1, ..., x_n) = c(X), i = 1, 2, ..., n, \]

And

\[ \left\{ \begin{array}{ll}
D v(x) = D z(x) + z(x) \tilde{a}'(x); & \text{if } n \text{ is even} \\
D v(x) = D z(x) - z(x) \tilde{a}'(x); & \text{if } n \text{ is odd}
\end{array} \right. \]

Using the fact that

\[ \left\{ \begin{array}{ll}
D z(x) \leq a(x)v(x) \tilde{a}'(x); & \text{if } n \text{ is even} \\
D z(x) \geq -a(x)v(x) \tilde{a}'(x); & \text{if } n \text{ is odd}
\end{array} \right. \]

From (3.11) and \( z(x) \leq v(x) \) from (3.12), we have

\[ \left\{ \begin{array}{ll}
D v(x) \leq [1 + a(x)]v(x) \tilde{a}'(x); & \text{if } n \text{ is even} \\
D v(x) \geq -[1 + a(x)]v(x) \tilde{a}'(x); & \text{if } n \text{ is odd}
\end{array} \right. \]

It is easy to estimate \( v(x) \) by following the same arguments as in the proof of Theorem 10 as follows

\[ v(x) \leq c(X) \exp \left[ \int_{\tilde{a}(x)}^{\infty} (1 + a(t)) dt \right]. \]

(3.13)
\[
\begin{align*}
-D_1 \cdots D_{n-1} z(x) &\leq c(X) \int_0^\infty a(x_1, \ldots, x_{n-1}, t_n) \exp \int_\tau^\infty (1 + a(\tau)) d\tau \\
&\quad \cdot dt_n \alpha_1'(x_1) \cdots \alpha_{n-1}'(x_{n-1}); \text{if } n \text{ is even} \\
-D_1 \cdots D_{n-1} z(x) &\geq -c(X) \int_0^\infty a(x_1, \ldots, x_{n-1}, t_n) \exp \int_\tau^\infty (1 + a(\tau)) d\tau \\
&\quad \cdot dt_n \alpha_1'(x_1) \cdots \alpha_{n-1}'(x_{n-1}); \text{if } n \text{ is odd}
\end{align*}
\]

By continuing this process, we obtain

\[
\begin{align*}
-D_1 z(x) &\leq c(X) \int_0^\infty \int_0^\infty a(x_1, t_2, \ldots, t_n) \exp \int_\tau^\infty (1 + a(\tau)) d\tau \\
&\quad \cdot dt_2 \alpha_1'(x_1); \text{if } n \text{ is even} \\
D_1 z(x) &\geq -c(X) \int_0^\infty \int_0^\infty a(x_1, t_2, \ldots, t_n) \exp \int_\tau^\infty (1 + a(\tau)) d\tau \\
&\quad \cdot dt_2 \alpha_1'(x_1); \text{if } n \text{ is odd}
\end{align*}
\]

By integrating the last inequality with respect to \(x_i\) from \(x_i\) to \(\infty\), we have

\[
z(x) \leq c(X) \exp \int_\infty^{\tilde{a}_i(x)} a(t) \exp \int_\tau^\infty (1 + a(\tau)) d\tau \, dt.
\]

Since \(X\) is arbitrary, by substituting the value of \(z(x)\) in (3.8), we obtain the inequality (3.6).

**Remark 8.** In the particular case when \(n = 2, x \in \mathbb{R}_+^2\), \((0, \omega) = (0, 0), \alpha_1(x_1) = x_1, \alpha_2(x_2) = x_2, \text{and } c(X) = c_1(x_1) + c_2(x_2)\) then theorem 11 reduces to Theorem 1 in [8]

**Corollary 12.** Let \(u(x), c(x), a(x), D_t u(x)\) and \(D_t u(x)\) be non-negative continuous functions for all \(i = 1, 2, ..., n\) defined for \(x \in \mathbb{R}_+^n\),

\[
limit_{x_i \to \infty} u(x_1, x_2, ..., x_n) = 0, \forall i = 1, 2, ..., n.
\]

Let \(\tilde{a} \in C(\mathbb{R}_+^n, \mathbb{R}_+^n)\) be non-decreasing functions in each variable, with \(\tilde{a}(x) \geq x \text{ on } \mathbb{R}_+^n\). Assume that \(c(x)\) is non-decreasing in each variable \(x \in \mathbb{R}_+^n\), if

\[
D_t u(x) \leq c(x) + M \left[ u(t) + \int_{\tilde{a}(x)}^\infty a(t) \{u(t) + D_t u(t)\} \, dt \right],
\]

for \(x \in \mathbb{R}_+^n\), then

\[
D_t u(x) \leq c(x) \exp \int_{\tilde{a}(x)}^\infty [M + a(t) + Ma(t)] \, dt.
\]

**Proof:** Fix any \(X \in \mathbb{R}_+^n\). Then, for \(x \leq X\) and from (3.15), we have

\[
D_t u(x) \leq c(X) + M \left[ u(t) + \int_{\tilde{a}(x)}^\infty a(t) \{u(t) + D_t u(t)\} \, dt \right],
\]

Define a function \(z(x)\) by

\[
z(x) = c(X) + M \left[ u(t) + \int_{\tilde{a}(x)}^\infty a(t) \{u(t) + D_t u(t)\} \, dt \right],
\]

then

\[
\lim_{x_i \to \infty} z(x_1, ..., x_n) = c(X), \quad i = 1, 2, ..., n,
\]

\[
D_t u(x) \leq z(x),
\]

By differentiating (3.8)

\[
D_t z(x) \leq M[a(x) + D_t u(x)] \{u(x) + D_t u(x)\} \tilde{a}'(x); \text{if } n \text{ is even}
\]

\[
D_t z(x) \geq -M[a(x) + D_t u(x)] \{u(x) + D_t u(x)\} \tilde{a}'(x); \text{if } n \text{ is odd}
\]

Using (3.18) and the fact that \(\Omega(x) \leq z(x)\), we have

\[
D_t z(x) \leq [M a(x) + a(x)] z(x) \tilde{a}'(x); \text{if } n \text{ is even}
\]

\[
D_t z(x) \geq -[M a(x) + a(x)] z(x) \tilde{a}'(x); \text{if } n \text{ is odd}
\]

Therefore

\[
z(x) \leq c(X) \exp \int_{\tilde{a}(x)}^\infty \left[ M + a(t) + Ma(t) \right] \, dt.
\]

Since \(X\) is arbitrary, by substituting the value of \(z(x)\) in (3.18) we obtain the inequality (3.16).

**Remark 9.** In the particular case when \(n = 2, x \in \mathbb{R}_+^2\), \((0, \omega) = (0, 0), \alpha_1(x_1) = x_1, \alpha_2(x_2) = x_2, \text{and } c(x) = c_1(x_1) + c_2(x_2)\) then corollary 12 reduces to theorem 2 in [7].

**Theorem 13.** Let \(u(x), c(x), a(x), b(x), f(x), D_t u(x)\) and \(D_t u(x)\) be non-negative continuous functions for all \(i = 1, 2, ..., \infty\), and \(\alpha_i(x) \Rightarrow a_i(x) \Rightarrow \tilde{a}_i(x)\) as \(x \to \infty\) for each \(i = 1, 2, ..., n\). If \(c(X) \leq c_0\) and \(\tilde{a}_i(x) \leq \tilde{a}_i(x)\) for all \(x \in \mathbb{R}_+^n\) and \(i = 1, 2, ..., n\). Then, for each \(x \in \mathbb{R}_+^n\), we have

\[
D_t u(x) \leq c(X) + M \int_{\tilde{a}(x)}^\infty a(t) \{u(t) + D_t u(t)\} \, dt.
\]
1, 2, ..., n defined for \( x \in \mathbb{R}^n_n \), and \( \bar{a} \in C^1(\mathbb{R}^n_n, \mathbb{R}^n) \) be non-decreasing functions in each variable, with \( \bar{a}(x) \geq x \) on \( \mathbb{R}^n_n \).

\[
\lim_{x_i \to \infty} u(x_1, x_2, ..., x_n) = 0, \forall i = 1, 2, ..., n.
\]

Let \( K(u(x)) \) be a real-valued, positive, continuous, strictly non-decreasing, sub-additive, and sub-multiplicative function for \( u(x) \geq 0 \), and \( H(u(x)) \) be a real-valued, positive, continuous and non-decreasing function defined for \( x \in \mathbb{R}^n_n \). Assume that \( c(x) \) and \( f(x) \) are non-decreasing functions in each of the variables \( x \in \mathbb{R}^n_n \). If

\[
D_i u(x) \leq c(x) + f(x)H \left( \int_{\bar{a}(x)}^{\infty} a(t)K(u(t))dt \right) + \int_{\bar{a}(x)}^{\infty} b(t)Du(t)dt,
\]

for all \( x \in \mathbb{R}^n_n \), then

\[
D_i u(x) \leq c(x) + f(x)H \left( G^{-1} \left[ \int_{\bar{a}(x)}^{\infty} a(t)K(f(t)p(t))dt \right] \right) \exp \int_{\bar{a}(x)}^{\infty} b(t)dt,
\]

for all \( x \in \mathbb{R}^n_n \), where

\[
p(x) = \int_{\bar{a}(x)}^{\infty} \left( \exp \int_{t}^{\infty} b(r)dr \right)dt.
\]

\[
\xi = \int_{0}^{\infty} a(t)K(c(t)p(t))dt.
\]

\[
G(r) = \int_{r}^{\infty} \frac{ds}{K(H(s))}.
\]

Where \( G^{-1} \) is the inverse function of \( G \), and

\[
G(\xi) + \int_{\bar{a}(x)}^{\infty} a(t)K(f(t)p(t))dt \in dom(G^{-1})
\]

for all \( x \in \mathbb{R}^n_n \).

Proof: Define a function \( z(x) \) by

\[
z(x) = c(x) + f(x)H \left( \int_{\bar{a}(x)}^{\infty} a(t)K(u(t))dt \right),
\]

then from (3.19), we have

\[
D_i u(x) \leq z(x) + \int_{\bar{a}(x)}^{\infty} b(t)Du(t)dt,
\]

Clearly, \( z(x) \) is a positive, continuous, and non-decreasing function in each of the variables \( x \in \mathbb{R}^n_n \). Using Theorem 10 in (3.25), we get

\[
Du(x) \leq z(x) \exp \left( \int_{\bar{a}(x)}^{\infty} b(t)dt \right).
\]

By integrating (3.26) with respect to \( x \) from \( x \) to \( \infty \), we obtain

\[
u(x) \leq z(x)p(x),
\]

where \( p(x) \) is defined in (3.21). From (3.24) we have

\[
z(x) = c(x) + f(x)H(v(x)),
\]

where

\[
v(x) = \int_{\bar{a}(x)}^{\infty} a(t)K(u(t))dt.
\]

Now, using (3.28) in (3.27) we get

\[
u(x) \leq [c(x) + f(x)H(v(x))]p(x),
\]

From (3.29) and (3.30) and since \( K \) is a sub-additive and sub-multiplicative function, we obtain

\[
v(x, y) \leq \int_{\bar{a}(x)}^{\infty} a(t)K(c(t)p(t))dt \leq \int_{\bar{a}(x)}^{\infty} a(t)K(c(t)p(t))dt + \int_{\bar{a}(x)}^{\infty} a(t)K(f(t)H(v(t))p(t))dt.
\]

Therefore

\[
v(x, y) \leq \int_{\bar{a}(x)}^{\infty} a(t)K(c(t)p(t))dt + \int_{\bar{a}(x)}^{\infty} a(t)K(f(t)H(v(t))p(t))dt.
\]

Define a function \( \Phi(x) \) by

\[
\Phi(x) = \int_{\bar{a}(x)}^{\infty} a(t)K(c(t)p(t))dt + \int_{\bar{a}(x)}^{\infty} a(t)K(f(t)H(v(t))p(t))dt.
\]

Then

\[
\lim_{x_i \to \infty} \Phi(x) = \int_{0}^{\infty} a(t)K(c(t)p(t))dt = \xi.
\]

And

\[
v(x) \leq \Phi(x).
\]

Clearly, \( \Phi(x) \) is a positive and non-decreasing function in each variable \( x_2, x_3, ..., x_n \), then

\[
D_i \Phi(x) = - \int_{a_2(x_2)}^{\infty} \int_{a_n(x_n)}^{\infty} a(\alpha_1(x_1), t_2, ..., t_n)K(f(\alpha_1(x_1), t_2, ..., t_n)p(\alpha_1(x_1), t_2, ..., t_n))dt_2 ... dt_n \alpha_1'(x_1).
\]

hence
From the inequalities Remark 10, we obtain the inequality

\[ D_t \Phi(x) \geq - \int a_1(x_1), t_2, ..., t_n)K \left( f(a_1(x_1), t_2, ..., t_n) \right) dt_2 ... dt_n a'_1(x_1) \]

i.e.

\[ \frac{D_t \Phi(x)}{K (H(\Phi(x)))} \geq - \int a_1(x_1), t_2, ..., t_n)K \left( f(a_1(x_1), t_2, ..., t_n) \right) dt_2 ... dt_n a'_1(x_1) \]

(3.33)

From (3.23) we have

\[ D_t G(\Phi(x)) = \frac{D_t \Phi(x)}{K (H(\Phi(x)))} \]

(3.34)

\[ D_t G(\Phi(x)) \geq - \int a_1(x_1), t_2, ..., t_n)K \left( f(a_1(x_1), t_2, ..., t_n) \right) dt_2 ... dt_n a'_1(x_1) \]

(3.35)

Now, by setting \( x_1 = t \) and integrating from \( x_1 \) to \( \infty \) in (3.35), and using (3.31) we get

\[ \Phi(x) \leq G^{-1} \left[ G(\xi) + \int_{\tilde{a}(x)} a(t)K(f(t)p(t))dt \right] \]

(3.36)

Finally, by substituting (3.28), (3.32) and (3.36), (3.34) we obtain the inequality (3.20).

**Remark 10.**

From the inequalities (3.30) and (3.36) in the proof of theorem 13, we can find this inequality

\[ u(x) \leq c(x) + f(x)H \left( G^{-1} \left[ G(\xi) + \int_{\tilde{a}(x)} a(t)K(f(t)p(t))dt \right] \right) \]

(3.37)

If we put \( n = 2, x, H(x) = x \), and \( a(x) = b(x) \) then Theorem 13 reduces to Theorem 1 in [7]

**Corollary 14.** Let \( u(x), c(x), a(x), b(x), D_t u(x) \) and \( D_t u(x) \) be non-negative continuous functions for all \( i = 1,2,...,n \) defined for \( x, \tilde{a} \in C^1(\mathbb{R}^+_n, \mathbb{R}^+_n) \) to be non-decreasing functions in each variable, with \( a(x) \) and \( b(x) \) be non-negative continuous functions for all \( i = 1,2,...,n \) defined for \( x, \tilde{a} \in C^1(\mathbb{R}^+_n, \mathbb{R}^+_n) \) to be non-decreasing functions in each variable, with \( a(x) \geq x \), and let

\[ \lim_{x \to +\infty} u(x_1, x_2, ..., x_n) = 0, \forall i = 1,2,...,n. \]

Corollary 15. Let \( u(x), a(x), b(x), D_t u(x) \) and \( D_t u(x) \) be non-negative continuous functions for all \( i = 1,2,...,n \) defined for \( x, \tilde{a} \in C^1(\mathbb{R}^+_n, \mathbb{R}^+_n) \) to be non-decreasing functions in each variable, with \( a(x) \) and \( b(x) \) be non-negative continuous functions for all \( i = 1,2,...,n \) defined for \( x, \tilde{a} \in C^1(\mathbb{R}^+_n, \mathbb{R}^+_n) \) to be non-decreasing functions in each variable, with \( a(x) \geq x \), and let

\[ \lim_{x \to +\infty} u(x_1, x_2, ..., x_n) = 0, \forall i = 1,2,...,n. \]
\[ Du(x) \leq M + \int_{\tilde{a}(x)}^{\infty} a(t)u(t)dt + \int_{\tilde{a}(x)}^{\infty} b(t)Du(t)dt, \]

for all \( x \in \mathbb{R}^n_+ \), where \( M > 0 \) is constant, then the following conclusion are true:

\[ Du(x) \leq M \left( 1 + \int_{0}^{\infty} a(t)p(t)dt \exp \int_{\tilde{a}(x)}^{\infty} a(t)p(t)dt \exp \int_{\tilde{a}(x)}^{\infty} b(t)dt, \right. \]

\[ u(x) \leq M \left( 1 + \int_{0}^{\infty} a(t)p(t)dt \exp \int_{\tilde{a}(x)}^{\infty} a(t)p(t)dt \right) p(x) \]

for \( x \in \mathbb{R}^n_+ \), where

\[ p(x) = \int_{\tilde{a}(x)}^{\infty} \left( \exp \int_{t}^{\infty} b(\tau)d\tau \right) dt. \]

Proof: By setting \( K(x) = x \) and \( c(x) = M \) in Corollary 14, we obtain the results of this Corollary.

**Corollary 16.** Let \( u(x), a(x), b(x), D_iu(x) \) and \( Du(x) \) be non-negative continuous functions for all \( i = 1, 2, ..., n \) defined for \( x \in \mathbb{R}^n_+ \), and \( \tilde{a} \in C(\mathbb{R}^n_+, \mathbb{R}^n_+) \) be non-decreasing functions in each variable, with \( \tilde{a}(x) \geq x \) on \( \mathbb{R}^n_+ \).

\[ \lim_{x_i \to \infty} u(x_1, x_2, ..., x_n) = 0, \forall i = 1, 2, ..., n. \]

Let \( K(u(x)) \) be a real-valued, positive, continuous, strictly non-decreasing, sub-additive, and sub-multiplicative function for \( u(x) \geq 0 \). If

\[ Du(x) \leq \sum_{i=1}^{n} c_i(x_i) + \int_{\tilde{a}(x)}^{\infty} a(t)K(u(t))dt + \int_{\tilde{a}(x)}^{\infty} b(t)Du(t)dt, \]

for all \( x \in \mathbb{R}^n_+ \), where \( c_i(x_i) > 0 \) and \( c_i'(x_i) \geq 0 \) are continuous functions for \( x_i \geq 0 \) for all \( i = 1, ..., n \) then

\[ Du(x) \leq \sum_{i=1}^{n} c_i(x_i) + \left( \int_{0}^{\infty} a(t)K(p(t))dt \exp \int_{\tilde{a}(x)}^{\infty} a(t)K(p(t))dt \right) \exp \int_{\tilde{a}(x)}^{\infty} b(t)dt, \]

for \( x \in \mathbb{R}^n_+ \), where where \( p(t) \) and \( T \) are defined in corollary 14, and

\[ \xi = \int_{\tilde{a}(x)}^{\infty} a(t)K(p(t)\sum_{i=1}^{n} c_i(t_i))dt. \]

**REFERENCES**