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First order autoregressive representation of Markov bi-dimensional chains of 1-order

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Abstract

This paper suggests an extension of Lai's results about the first order autoregressive representation of Markov bi-dimensional chain of 1-order. In the case of markov chain with independent components, we find of course the conditions validating these results for each component.

Keywords: *Markov'chains , autoregressive process, spectral density, diagonal development of bivariate distribution*

Résumé

Ce papier suggère une extension d'un résultat de Lai à propos de la représentation autorégressive des chaînes de Markov bi-dimenssionnelle d'ordre 1. Dans le cas où les composantes de la chaîne sont indépendantes, nous retrouvons naturellement les conditions validant ces résultats pour chaque composante.

Mots clés: *Chaînes de Markov, processus autoregressif, densité spectrale, développement diagonal des distributions bivariées.*

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ملخص

هـــدا العمـــل يقترح تعميم نتائج LAI التي تخص بتمثيـــل AR (1) للسلاســـل Markov من الدرجة 1. في حالة سلسلـــة Markov دات مركبات مستقلة نتحصل طبعا على الشروط LAI متحققة بالنسبة لكل مركبة

الكلمات المفتاحيـــة: تمثيـــل AR (1) للسلاســـل سلسلـــة Markov

Generally, a markov chain is not an autoregressive process, except for first order homogeneous, aperiodic and irreducible markov chains at two states $\{0,1\}$. However, Lai (1977) has shown that under convenient hypothesis, a first order uni-dimensional markov chain, valued in a finite or non-finite countable set, belong to a first order autoregressives processes class. In this paper, we extend this result to the cases of first order bi-dimensional markov chains valued in $\{0,1,\ldots,N\}^2$. Firstly, we study Lai's analogue conditions and secondly the case of the bivariate distributions having a diagonal development.

1. AR(1) representation of first order bi-dimensional markov chains

Let $(Z_t = (X_t, Y_t) : t \in Z)$ be a first order bi-dimensional markov chain valued in $\{0,1,...,N\}^2$. If the chain (Z_t) is stationary, the probabilities $p_{x,y}(t) = P(Z_t = (x,y))$ and the transition probabilities $p_{x,y,x',y'}(t) = P(Z_{t+1} = (x',y') / Z_t = (x,y))$ are independent of time t. Let's set down P as the transition probabilities matrix, P^n its n^{th} power $(n \geq 1)$ and $\pi = ((\pi_{x,y}) \ : \ 0 \leq x,y \leq N)$ its stationary distribution.

We start writing the covariance function γ (.) of the chain (Z_t) under the form of a matrix product. After that we determine its spectral density which we shall compare to that of autoregressive process.

Proposition 1

Let $(Z_t = (X_t, Y_t) : t \in Z)$ be a first order bi-dimensional markov chain valued in $\{0,1,...,N\}^2$. If this chain is stationary, then its covariance matrix γ (.) has the form

$$\gamma(n) = A\left[P^n - 1_{(N+1)^2}.\Pi\right]B, \quad \forall n \geq 0 \qquad (1.1)$$

where $\Pi = (\pi_{.0}, \pi_{.1}, \ldots, \pi_{.N})$ with $\pi_{.j} = (\pi_{0j}, \ldots, \pi_{Nj})$, $1_{(N+1)^2} = {}^t(1, \ldots, 1)$ a vector of $R^{(N+1)^2}$ and A and B are conveniently chosen matrices.

Proof. Let us set down that the relation (1.1) determines entirely the covariance function $\gamma(.)$ and a simple computation provides us the coefficients of the covariance matrix $\gamma(n)$:

$$\sigma_{i,j}(n) = \begin{cases} \sum\limits_{x,y,x',y'}^{N} x^{2-j} y^{j-1} (x')^{2-i} (y')^{i-1} . \pi(x,y) . p^{(n)}{}_{x,y,x',y'} - \alpha_{i,j}\beta_{i,j} & if \quad n > 0 \\ \\ \sum\limits_{x,y,x',y'}^{N} x^{4-(i+j)} y^{i+j-2} . \pi(x,y) - \alpha_{i,j}\beta_{i,j} & if \quad n = 0 \end{cases}$$

with

$$\alpha_{i,j} = \left(\sum_{x,y=0}^{n} x.\pi(x,y)\right)^{4-(i+j)}$$

and

$$\beta_{i,j} = \left(\sum_{x,y=0}^{n} y.\pi(x,y)\right)^{i+j-2}, \quad \forall i, j = 1,2$$

Considering then the matrix

$$A = \begin{pmatrix} A_{0.} & A_{1.} & . & . & . & A_{N.} \\ A'_{0.} & A'_{1.} & . & . & . & A'_{N.} \end{pmatrix}$$

where

$$A_{x.} = x.(\pi(x,0) \quad \pi(x,1) \quad . \quad . \quad . \quad \pi(x,N)),$$

$$A'_{x.} = (0, \pi(x,1), 2\pi(x,2), \ldots, N\pi(x,N))$$

and the matrix

$$B = {}^t\begin{pmatrix} 0.{}^t1_{N+1} & 1.{}^t1_{N+1} & 2.{}^t1_{N+1} & . & . & . & N.{}^t1_{N+1} \\ (0,1,\ldots,N) & (0,1,\ldots,N) & (0,1,\ldots,N) & . & . & . & (0,1,\ldots,N) \end{pmatrix}$$

we verify easily that

$$A.\left(P^n - 1_{(N+1)^2}.\Pi\right).B = \gamma(n)$$

This is being true for every $n \geq 0$, setting down of course $P^0 = I_{(N+1)^2}$ the (N+1)²-order unit matrix.

Proposition 2

Let $(Z_t : t \in Z)$ be a first order bi-dimensional markov chain with valued in $\{0,1,...,N\}^2$, irreducible, aperiodic and stationary and let be its stationary distribution. If the matrix P of transition probabilities of $(Z_t : t \in Z)$ admits simple and non nul eigenvalues and if one at least of the following conditions (C) is satisfied, then the spectral density of the chain $(Z_t : t \in Z)$ is given by :

$$f(\lambda) = \frac{1}{2\pi} \frac{1 - z_2{}^2}{1 + z_2{}^2 - 2z_2.\cos(\lambda)} \gamma(0) \qquad (1.2)$$

where z_2 is the secund large eigenvalue (in absolute value) of P

Conditions (C) :

$$i) \begin{cases} \sum\limits_{x,y=0}^{N} x.\pi(x,y+1).u^{(j)}{}_{x(N+1)+y+1} = 0 \\ and \\ \sum\limits_{x,y=0}^{N} y.\pi(x,y+1).u^{(j)}{}_{x(N+1)+y+1} = 0 \end{cases}, \quad ii) \begin{cases} \sum\limits_{x,y=0}^{N} x.v^{(j)}{}_{x(N+1)+y+1} = 0 \\ and \\ \sum\limits_{x,y=0}^{N} y.v^{(j)}{}_{x(N+1)+y+1} = 0 \end{cases}$$

$\forall j = 3, \ldots, (N+1)^2$ *and where* $u^{(j)}$ *and* $v^{(j)}$ *are respectively the eigenvectors of matrices* P *and* ${}^tP$ *associated to the eigenvalue* z_j.

Proof.

Taking into account the factorization (1.1) of $\gamma(n)$ and that $I - 1_{(N+1)^2}.{}^t\Pi = \gamma(0)$ is symetrical, we can write the series $\sum_{n\in Z} e^{-in\lambda}\gamma(n)$ under the form

$$\sum_{n\in Z} e^{-in\lambda}\gamma(n) = -A.\left[I - 1_{(N+1)^2}.\Pi\right]B + A.G\left(e^{-i\lambda}\right)B + {}^t\left[A.G\left(e^{i\lambda}\right)B\right] \quad (1.3)$$
$$- \frac{1}{1-e^{i\lambda}} A.1_{(N+1)^2}.\Pi.B - \frac{1}{1-e^{-i\lambda}}{}^t\left[A.1_{(N+1)^2}.\Pi.B\right]$$

where $G(z) = (I - z.P)^{-1}$: $|z| \le 1$ is the generating function of the transition probabilities matrix P. As P is supposed to be simple and its eigenvalues aren't nul, then following result of Lancaster (1968) and taking into account (C) conditions, we can write :

$$A.G(z).B = -\frac{1}{\bar{z}-1} A.u_1.{}^tv_1.B - \frac{\lambda_2}{\bar{z}-\lambda_2} A.u_2{}^tv_2B$$

and

$${}^t\left[A.G(z).B\right] = -\frac{1}{z-1}{}^t\left(A.u_1.{}^tv_1.B\right) - \frac{\lambda_2}{z-\lambda_2}{}^t\left(A.u_2{}^tv_2B\right)$$

where $u_1 = 1_{(N+1)^2}$ *and* $v_1 = \Pi$ are the eigenvectors of P and ${}^tP$ associated to the eigenvalue $z_1 = 1$ *and* λ_2 indicate the inverse of the eigenvalue z_2 (we shall notice that the vector is solution of the equation $\Pi = \Pi.P$ because π is a stationary distribution). Reporting these expressions in (1.3), this latter becomes

$$\sum_{n\in Z} e^{-in\lambda}\gamma(n) = -A\left[I - 1_{(N+1)^2}.{}^t\Pi\right]B - \frac{1}{1-e^{-i\lambda}} Au_1.{}^tv_1.B - \frac{1}{1-e^{i\lambda}}{}^t\left(Au_1.{}^tv_1.B\right)$$
$$-\frac{\lambda_2}{e^{-i\lambda}-\lambda_2} Au_2.{}^tv_2.B - \frac{\lambda_2}{e^{-i\lambda}-\lambda_2}{}^t\left(Au_2.{}^tv_2.B\right) \quad (1.4)$$

The sum of the first three terms on the right hand side of (1.4) is reduced to $-AB$ (because the matrices AB and $A.1_{(N+1)^2}.{}^t\Pi.B$ are symetrics), and taking into account (C) conditions, we have (Lancaster (1968)) :

$$Au_2{}^tv_2B = AB - A.1_{(N+1)^2}.{}^t\Pi.B$$

Substiting these results in relation (2.4), we obtain :

$$\sum_{n\in Z} \gamma(n).\bar{z}^n = \frac{\lambda_2^2 - 1}{|z-\lambda_2|^2} Au_2{}^tv_2B$$

from which we deduce the result of the proposition.

Now, we deduce the autoregressive representation of the chain $(Z_t : t \in Z)$.

Corollary 1

Let $(Z_t : t \in Z)$ *be a first order bi-dimensional markov chain with states in* $\{0,1,\ldots,N\}^2$.

If this chain satisfies the conditions in the proposition 2, then this chain is a first order autoregressive and admits as representation

$$Z_t = z_1.Z_{t-1} + \varepsilon_t \quad (1.5)$$

where $(\varepsilon_t ; t \in Z)$ *is a sequence of an uncorrelated random variable, having as covariance matrix*

$$\Gamma_1 = \left(1 - z_1^2\right).\gamma(0).$$

The representation (1.5) allows us to see that the components X_t *and* Y_t of the chain (Z_t) satisfy a first order autoregressive equation without being markov chains. This result is then legitimely proved by the fact that if the components (X_t) *and* (Y_t) are stochastically independent, so each component defines a first order markov chain valued in $\{0,1,\ldots,N\}^2$, which are homogeneous, aperiodic and irreducible, and if P_1 *and* P_2 are the transition probabilities matrices of X_t *and* Y_t respectively, then we have $P = P_1 \otimes P_2$ (tensorial product of P_1 *and* P_2) and it is easy to see that the covariance matrix R is in this case $\begin{pmatrix} \gamma_1 & 0 \\ 0 & \gamma_2 \end{pmatrix}$ where γ_1 *and* γ_2 are the covariance matrices of (X_t) *and* (Y_t) respectively. On the other hand, if λ is an eigenvalue of P_1 *and* ${}^t(x_1, \ldots, x_{N+1})$ is an eigenvector associated to λ, then the latter is also eigenvalue of P associated to the eigenvector $u = {}^t(u_1, \ldots, u_{N+1})$ with $u_j = {}^t(x_1, \ldots, x_{N+1})$: $\forall j = 1, \ldots, N$, and if λ is an eigenvalue of P_2 and ${}^t(y_1, \ldots, y_{N+1})$ is an eigenvector associated to λ, then so λ is also an eigenvalue of P associated to the eigenvector $\omega = {}^t(\omega_1, \ldots, \omega_{N+1})$ with $\omega_j = y_j.1_{N+1}$: $\forall j = 1, \ldots, N+1$. From this, we show that the conditions (C) are reduced to Lai's conditions for each chain (X_t) *and* (Y_t). This confirms the individual autoregressive representation of (X_t) *and* (Y_t) deduced from the equation (1.5).

02. Diagonal Development

According once more to Lai, let's suppose that the transition probabilities of a bi-dimensional markov chain (Z_t) satisfy the relation

$$p_{x,y,x',y'} = p(x',y')\sum_{n,m=0}^{N}\rho_{n,m}.\theta_{n,m}(x,y).\theta_{n,m}(x',y') \qquad (2.1)$$

where $.(\theta_{n,m})$: $n,m\in\{0,....,N\}$ is a sequence of orthogonal functions relatively to the law $\{p(x,y)=P(X_t=x,Y_t=y)\ :\ x,y\in\{0,....,N\}\}$

$$\left(i.e.\ \sum_{n,m=0}^{N} p(x,y).\theta_{n,m}(x,y).\theta_{n',m'}(x,y)=\delta_{n,n'}.\delta_{m,m'},\quad \delta_{i,j}\ \textit{is kronecker's symbol}\right)$$

and $(\rho_{n,m})$ is a sequence of parameters caracterizing the bivariate distribution of (Z_t) and (Z_{t-1}) which we find directly from (1.4). We find the following result :

Lemma 1

If the markov chain (Z_t) is irreducible and if we suppose $\rho_{00}=\rho_{10}=\rho_{01}=1,\quad \theta_{00}(x,y)=1,$ $\theta_{10}(x,y)=\alpha_{10}x+\beta_{1,0}$ *and* $\theta_{01}(x,y)=\alpha_{01}x+\beta_{01}, \forall x,y\in\{0,1,....,N\}$ *and* *where* α_{10} *and* α_{01}, β_{10} *and* β_{01} *are real constants with* $\alpha_{10}\neq 0$ *and* $\beta_{10}\neq 0$, *then we have*

$$E(Z_t.\theta_{n,m}(Z_t))=0\ ;$$

$$\forall(n,m)\in\{0,1,....,N\}^2-\{(0,0),(1,0),(0,1)\}.$$

If the components of the chain are independent, with the same initial law and for each transition probability admitting a diagonal development, then the chain (Z_t) admits an ease to write diagonal development. In reverse, if the transition probabilities of the bivariate chain admit a diagonal development and is at independent components and if $\theta_{n,m}(x,y)=\theta_n(x).\theta_m(y)$ and $\rho_{n,m}(x,y)=\rho_n(x).\rho_m(y)$, then the transition probabilities of each chain (X_t) *and* (Y_t) admit a diagonal development.

A markov bi-dimensional chain of which the transition probabilities are of the form (2.1) is necessarily stationary.

Lemma 2

The eigenvalues of P are $(\rho_{n,m}:n,m=0,...,N)$ with for eigenvectors on the right $\theta_{n,m}=(\tilde\theta_{n,m}(0),...,\tilde\theta_{n,m}(N))$ *where* $\tilde\theta_{n,m}(j)=(\theta_{n,m}(0,j),....,\theta_{n,m}(N,j))$ *and as eigenvectors on the left*

$${}^tQ_{n,m}=(\tilde Q_{n,m}(0),...,\tilde Q_{n,m}(N))$$

where $\tilde Q_{n,m}(j)=(Q_{n,m}(0,j),....,Q_{n,m}(N,j))$ *with* $Q_{n,m}(x,y)=p(x,y).\theta_{n,m}(x,y);\ \forall(x,y)\in\{0,1,....,N\}^2$. Moreover

$$\sum_{n,m=0}^{N}\theta_{n,m}.{}^tQ_{n,m}=I_{(n+m)^2}.$$

Proof.

Let's set down $(p_{x,y,x',y'}:0\le x,x',y,y'\le N)$ the transition probabilities matrix and note $\Pi_{n,m}$ the $(n,m)^{th}$ row :

$$\Pi_{n,m}=(p_{n,m,0,0},...,p_{n,m,N,0},p_{n,m,0,1},...,p_{n,m,N,1},\ldots\ldots,p_{n,m,0,N},...,p_{n,m,N,N})$$

By using diagonal development (3.1) of the transition probabilities $p_{n,m,x,y}$ and orthogonality of the sequence $(\theta_{n,m}(.,.)\ :\ 0\le n,m\le N)$, we easily set up without any difficulty that $\rho_{n,m}$ is an eigenvalue of P associated to the eigenvector $\theta_{n,m}$. Similarly, we check that $Q_{n,m}$ is an eigenvector of the matrix ${}^tP$ associated to the eigenvalue $\rho_{n,m}$. Let's show now that $\theta_{n,m}$ satisfies the dual orthogonality relationship. Let's note Θ the eigenvectors matrix, $\theta_{n,m}$, and let Θ^{-1} be its inverse. Let's set down

$$\theta(x,y)=\begin{pmatrix}\theta_{0,0}(x,y),...,\theta_{N,0}(x,y),\theta_{0,1}(x,y),...,\theta_{N,1}(x,y),\ldots\\ \ldots,\theta_{0,N}(x,y),...,\theta_{N,N}(x,y)\end{pmatrix}$$

the $(x,y)^{th}$ row vector of Θ and

$$\alpha(x,y)={}^t\begin{pmatrix}\alpha_{0,0}(x,y),...,\alpha_{N,0}(x,y),\alpha_{0,1}(x,y),..\\ .,\alpha_{N,1}(x,y),\ldots\ldots,\alpha_{0,N}(x,y),...,\alpha_{N,N}(x,y)\end{pmatrix}$$

the $(x,y)^{th}$ column vector of Θ^{-1}. We have

$$\theta(x,y).\alpha(x',y')=\sum_{n,m=0}^{N}\theta_{n,m}(x,y).\alpha_{n,m}(x',y')=\delta_{x,x'}.\delta_{y,y'}$$

Since the sequence $(\theta_{n,m}(.,.))$ is $p(.,.)$-orthogonal, then the coefficients $\alpha_{n,m}(x',y')$ are provided by

$$\alpha_{n,m}(x',y')=p(x',y').\theta_{n,m}(x',y')$$

for every x', y', and so,

$$\theta_{n,m}(x,y).\theta_{n,m}(x',y').\alpha_{n,m}(x',y') =$$

$$\sum_{n,m=0}^{N}\sum_{n,m=0}^{N}\frac{\theta_{n,m}(x,y).\alpha_{n,m}(x',y')}{p(x',y')} = \frac{\delta_{x,x'}.\delta_{y,y'}}{p(x',y')}$$

Now, the $(x,y)^{th}$ coefficient of $\Pi_{n,m}$ is $\theta_{n,m}(x,y).Q_{n,m}(x',y').p(x',y')$, then

$$\sum_{n,m=0}^{N}\theta_{n,m}.{}^{t}Q_{n,m} = \sum_{n,m=0}^{N}\Pi_{n,m} = I_{(n+m)^2}.$$

Theorem

Let (Z_t) be an irreducible markov chain for which the transition probabilities $p_{x,y,x',y'}$ satisfy the diagonal development (2.1). If the eigenvalues $\rho_{n,m}$ are simple and real and $\rho_{0,0}=1$ and if $\begin{cases}\theta_{1,0}(x,y)=\alpha_{1,0}.x+\beta_{1,0}\\ \theta_{0,1}(x,y)=\alpha_{0,1}.x+\beta_{0,1}\end{cases}$*, then the process (Z_t) is a first order autoregressive process.*

Proof.

The required conditions in the lemma 1 and 2 being satisfied, then the (C) conditions of the proposition 2 are also satisfied, and consequently the chain (Z_t) is indeniably a first order autoregressive process.

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