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PATH INTEGRAL
FOR FREE RELATIVISTIC SPINNING PARTICLE

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Abstract

The purpose of this paper is to investigate the path integral representation of Dirac propagator by considering the case of the free spinning particle. Thanks to the introduction of a suitable transformation, the domain of integrations over Grassmannian variables becomes free from the restriction and then via the Grassmannian sources, the integration over relative velocities are readily carried out. Such a way of calculation allows us to get the explicit spinor structure of the propagator.

***Key words**: Path-integral, Dirac particle, relativistic equation.*

Résumé

Le but de ce papier est d'investiguer le formalisme des intégrales de chemins pour le propagateur de Dirac en considérant le cas de la particule libre. Grâce à une transformation convenable, les intégrations sur les variables de Grassmann ne sont plus restreintes et l'introduction des sources Grassmanniennes fait que les intégrations sur les vitesses sont facilement effectuées. Ces calculs, nous ont permit d'avoir une forme spinorielle explicite du propagateur.

***Mots clés**: Intégral de chemin, particule de Dirac, équation relativistique.*

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ملخص

الهدف من هذا المقال هو استخدام التكاملات على المسلك لعرض منتشر ديراك و تطبيقه على حالة الجسيم الحر.
بالإستعانة بتحويلات مناسبة فإن التكاملات على متغيرات غراسمان تصبح لا شرطية. ادخال منابع غراسمان تجعل التكاملات على السرعة سهلة.
إن هذه الطريقة تسمح لنا بالحصول على بنية واضحة للمنتشر النسبي.

***الكلمات المفاتيح**: التكاملات علىالمسلك، جسيم ديراك، المعادلة النسبية.*

Many different attempts to incorporate spin into a path integral formulation have been proposed in the past. However, because the spin takes discrete values it has been difficult to suggest for it a continuous path. Nevertheless, there has been a few approaches proposed to solve this complicate problem which can classify in two categories: either the spin is described by bosonic (commuting) variables, or by fermionic (anticommuting) variables. In the bosonic case we can quote firstly the early model suggested by Feymann who has proposed a path integral representation of the free Dirac electron in one dimension based on Poisson stochastic process [1]. However, its extension to the case of interaction has not been investigated. The second attempt is the Schulman where he has tried to describe this spin of the particle as a top [2]. Unfortunately, its extension to the relativistic spinning particle was faced to serious difficulties due to the complicated dynamics of the relativistic top. The last model of the first category is the Barut one [2], where the spin evolution is described by internal variables (c-number) different of those of the top. The second category, due to Berezin and Marinov [4], could be considered as a renew of the Dirac propagator by means of path integral in terms of Grassmannian variables, of the exp($i$action). One can claim that this was the first successful attempt, which dares to describe correctly the spinning point particle. In the last decade, Fradkin and Gitman [2] have returned to this model and have succeeded to establish a rigorous formulation of this path integral representation with effective classical actions being already reparametrization invariant and super gauge invariant. Besides, the interest in this model is its close relationship with string theory.

Our paper is organized as follow. Section 1, we present a review of the path integral representation for the relativistic spin ½ particle propagator, derived by Fradkin and Gitman [2] where the spinning degrees of freedom are described by Grassmannian variables. In section 2, we apply this formalism to the fundamental case of the free particle where we carried out explicitly all bosonic and Grassmannian integrations. To this aim, we use a convenient transformation, which liberates the spin variables from the boundary conditions. That is the spin Gaussian integrations are not restricted and the integration over Grassmannian proper time allows us to extract the so-called Polyakov spin factor.

RELATIVISTIC SPINNIG PARTICLE

In this section, we present a review of the path integral representation for the relativistic spinning particle propagator, constructed and investigated by Fradkin and Gitman [2]. The main interest of this representation consists in its reparametrization invariant and super gauge invariant action.

Consider first a relativistic spinning particle interacting with an external electromagnetic field described by a vector potential $A^\mu(x)$. The corresponding causal Green's function $S^c(x,y)$ satisfies to the Dirac equation:

$$\left(P_\mu \gamma^\mu - m\right) S^c(x,y) = -\delta(x-y), \quad (1)$$

where:

$$P_\mu = i\partial_\mu - gA_\mu . \quad (2)$$

Multiplying both sides of Eq. (1) by γ^5, we get:

$$(P\tilde{\gamma} - m)\tilde{S}^c(x,y) = \delta(x-y), \quad (3)$$

where $\tilde{S}^c(x,y) = S^c(x,y)\gamma^5$ and $\tilde{\gamma}^\mu = \gamma^5\gamma^\mu$, $(\gamma^5)^2 = -1$.

Following Schwinger proper time method, the propagator $\tilde{S}^c(x,y)$ is treated as a matrix element of an operator $\tilde{S}^c$ in coordinate space:

$$\tilde{S}^c(x,y) = \langle x|\tilde{S}^c|y\rangle . \quad (4)$$

Making use of Eq.(3) and Eq.(4) it follows that:

$$(P\gamma - m\gamma^5)\tilde{S}^c = I. \quad (5)$$

So the operator $\tilde{S}^c$ is the inverse of the pure Fermi operator $(P\gamma - m\gamma^5)$. Consequently it can be presented by means of integration over even and odd Grassmann variables:

$$\tilde{S}^c = \int_0^\infty d\lambda \int \exp\left\{i\left[\lambda(P\gamma - m\gamma^5)^2 + i\varepsilon\right] + i\chi(P\gamma - m\gamma^5)\right\} d\chi \quad (6)$$

where λ is an even and χ is an odd variable. Using Eq.(4) we get:

$$\tilde{S}^c(x_b, x_a) = \int_0^\infty d\lambda \int \langle x_b | \exp(-iH) | x_a \rangle d\chi . \quad (7)$$

where $H=\lambda[m^2-(P\gamma)^2]+(P\gamma-m\gamma^5)\chi$. Now in order to obtain a path integral representation for the propagator, we fellow the usual procedure by dividing the time interval of the evolution into N equal parts and then inserting $(N-1)$ times the unit decompositions $\int |x\rangle\langle x| dx$, we obtain:

$$\tilde{S}^c(x_b, x_a) = \lim_{N\to\infty} \int_0^\infty d\lambda_0 \int d\chi_0 \int dx_1 \ldots dx_{N-1} d\lambda_1 \ldots d\lambda_N d\chi_1 \ldots d\chi_N \times \prod_{k=1}^N \delta(\lambda_k - \lambda_{k-1})\delta(\chi_k - \chi_{k-1}) \langle x_k | \exp[-iH(\lambda_k, \chi_k)\Delta\tau] | x_{k-1} \rangle, \quad (8)$$

where $x_a=x_0$, $x_b=x_N$ and $\Delta\tau=1/N$.

Expanding the matrix elements, as usual, to the first order in $\Delta\tau$ and then inserting the resolutions of unity $\int |p\rangle\langle p| dp$, the Green's function is written, using the midpoint prescription, as:

$$\tilde{S}^c(x_b, x_a) = \lim_{N\to\infty} \int_0^\infty d\lambda_0 \int d\chi_0 \int dx_1 \ldots dx_{N-1} \frac{dp_1}{(2\pi)^4} \cdots \frac{dp_N}{(2\pi)^4} \times d\lambda_1 \ldots d\lambda_N \int d\chi_1 \ldots d\chi_N \prod_{k=1}^N \left\{ \exp[ip_k \frac{\Delta x_k}{\Delta\tau} - iH(\lambda_k, \chi_k, \bar{x}_k, p_k]\Delta\tau \right\} \times \delta(\lambda_k - \lambda_{k-1})\delta(\chi_k - \chi_{k-1}). \quad (9)$$

Using the integral representation of delta functions:

$$\delta(\lambda) = \int \frac{dp_\lambda}{2\pi} \exp(ip_\lambda \lambda) \text{ and } \delta(\chi) = i\int \frac{dp_\lambda}{2\pi} \exp(ip_\chi \chi), \quad (10)$$

we get:

$$\tilde{S}^c(x_b, x_a) = \mathbf{T}\int_0^\infty d\lambda_0 \int d\chi_0 \int DxDpD\lambda Dp_\lambda D\chi Dp_\chi \times \exp\left\{ i\int \left[\lambda(P^2 - m^2 - i\frac{g}{2}F_{\alpha\beta}\gamma^\alpha\gamma^\beta + \left(m\gamma^5 - P_\mu\gamma^\mu\right)\chi + p\dot{x} + p_\lambda\dot{\lambda} + p_\chi\dot{\chi} \right] d\tau \right\} \quad (11)$$

with $\mathbf{T}$ stants for the Dyson time ordering symbol, necessary in this case because γ matrices are supposed formally as dependent on time τ. This allows us to deal with γ matrices like with odd Grassmann variables, consequently one can associate to $\gamma^n(\tau)$ five odd sources $\rho^n(\tau)$ such that:

$$\mathbf{T}\exp\left\{F(\gamma^n(\tau))\right\} = \exp\left\{F(\frac{\delta_l}{\delta\rho_n})\right\} \mathbf{T}\left\{\int_0^1 \rho_n\gamma^n d\tau \;\Big|_{\rho=0}\right\} \quad (12)$$

Next, one can present the quantity $\mathbf{T}\exp\left\{\int_0^1 \rho_n\gamma^n d\tau\right\}$ via a Grassmannian path integral:

$$\mathbf{T}\exp\left\{\int_0^1 \rho_n\gamma^n d\tau\right\} = \exp\left(i\gamma^n \frac{\partial_l}{\partial\theta^n}\right) \times \int D\psi \exp\left\{\int_0^1 (\psi\dot{\psi} - 2i\rho_n\psi^n)d\tau + \psi_n(1)\psi^n(0)\right\}, \quad (13)$$

with:

$$D\psi = D\psi\left[\int D\psi \exp\left\{\int_0^1 \psi\dot{\psi} d\tau\right\}\right]^{-1}, \quad (14)$$

where θ^n are odd variables anticommuting with γ matrices and ψ^n are odd variables obeing to the boundary conditions

$$\psi^n(1) + \psi^n(1) = \theta^n . \quad (15)$$

Using Eq.(13) we get the Hamiltonian path integral representation of the Green's function:

$$\tilde{S}^c(x_b, x_a) = \exp\left(i\gamma^n \frac{\partial_l}{\partial\theta^n}\right) \int_0^\infty d\lambda_0 \int d\chi_0 \int DxDpD\lambda Dp_\lambda D\chi Dp_\chi D\psi \times \exp\left\{ i\int \left[\lambda\left(P^2 - m^2 + 2igF_{\mu\nu}\psi^\mu\psi^\nu\right) - 2i\left(m\psi^5 - P_\mu\psi^\mu\right)\chi - i\psi\dot{\psi} + p\dot{x} + p_\lambda\dot{\lambda} + p_\chi\dot{\chi} \right] d\tau + \psi_n(1)(1)\psi^n(0) \right\}\Big|_{\theta=0} . \quad (16)$$

To obtain the Lagrangian formulation, we make the shift $p^{\mu} \to -p^{\mu} - \frac{\dot{x}}{e} - gA^{\mu}(x)$, where e=2λ. Then, we have:

$$\widetilde{S}^c(x_b, x_a) = \exp\left(i\gamma^n \frac{\partial_l}{\partial\theta^n} \right) \int_0^\infty de_0 \int d\chi_0 \int \exp\left\{ i\int_0^1 \left[-\frac{\dot{x}^2}{2e} - \frac{e}{2}m^2 - g\dot{x}A(x) + iegF_{\mu\nu}(x)\psi^\mu\psi^\nu + i\left(\frac{\dot{x}_\alpha\psi^\alpha}{e} - m\psi^5 \right)\chi - i\psi_n\dot{\psi}^n + p_\chi\dot{\chi} + p_e\dot{e} \right] d\tau + \psi_n(1)\psi^n(0) \right\}_{\theta=0} M(e) DxDeDp_e D\chi Dp_\chi D\psi, \quad (17)$$

where $M(e) = \int Dp\, exp\left\{ \frac{1}{2}\int_0^1 ep^2 d\tau \right\}$ is the normalization measure. As we will see in what follows, this measure will play the role of absorbing a divergency which comes from the linearization of the quadratic term in the variable x (the kinetic term) present in the action. It should be noted that the boundary term $\psi_n(1)\psi^n(0)$ and the antiperiodic conditions for the spin variables Eq.(15) in this approach arise by natural way.

FREE DIRAC PARTICLE PROPAGATOR

In order to develop some techniques with the path integral representation Eq.(17), let us consider in detail the simplest and fundamental case of a free particle, for which the free Dirac propagator $\widetilde{S}^c(x_b, x_a)$ is given by Eq.(17) with vanishing electromagnetic field $A(x)$:

$$\widetilde{S}_0^c(x_b, x_a) = \exp\left(i\gamma^n \frac{\partial_l}{\partial\theta^n} \right) \int_0^\infty de_0 \int d\chi_0 \int \exp\left\{ i\int_0^1 \left[-\frac{\dot{x}^2}{2e} - \frac{e}{2}m^2 + i\left(\frac{\dot{x}_\alpha\psi^\alpha}{e} - m\psi^5 \right)\chi - i\psi_n\dot{\psi}^n + p_\chi\dot{\chi} + p_e\dot{e} \right] d\tau + \psi_n(1)\psi^n(0) \right\}_{\theta=0} M(e) DxDeDp_e D\chi Dp_\chi D\psi, \quad (18)$$

Let us first fix the gauge conditions by performing functional integrations over p_e and p_χ which produce respectively the delta functional $\delta(\dot{e})$ and $\delta(\dot{\chi})$. These delta functions remove the integrations over the paths e and χ by fixing them to

$$e = e_0\ ,\ \chi = \chi_0. \quad (19)$$

Completing the square in $\dot{x}$ and linearizing it, we get:

$$\widetilde{S}_0^c(x_b, x_a) = \exp\left(i\gamma^n \frac{\partial_l}{\partial\theta^n} \right) \int_0^\infty de_0 \int d\chi_0 \int \exp\left\{ i\int_0^1 \left[p\dot{x} + \frac{e_0}{2}\left(p^2 - m^2\right) - i\left(\psi p + m\psi^5\right)\chi_0 - i\psi_n\dot{\psi}^n \right] d\tau + \psi_n(1)\psi^n(0) \right\}_{\theta=0} DxDpD\psi. \quad (20)$$

If now we integrate the first term present in the action by parts to isolate x and next perform the functional integral over x we get $\delta(\dot{p})$ which implies that

$$p = Const. \quad (21)$$

p is, as it is expected, a constant four vector. This reflects the conservation of the momentum. After disentangling the delta functions only one integration over p remains and the result will be:

$$\widetilde{S}_0^c(x_b, x_a) = \exp\left(i\gamma^n \frac{\partial_l}{\partial\theta^n} \right) \int_0^\infty de_0 \int d\chi_0 \frac{dp}{(2\pi)^4} \exp\left\{ ip(x_b - x_a) + i\frac{e_0}{2}\left(p^2 - m^2\right) \right\} \int D\psi \exp\left\{ \int_0^1 [\left(\psi p + m\psi^5\right)\chi_0 + \psi_n\dot{\psi}^n] d\tau + \psi_n(1)\psi^n(0) \right\}_{\theta=0}. \quad (22)$$

In the next step we integrate out the Grassmann proper time χ_0 and obtain:

$$\widetilde{S}_0^c(x_b, x_a) = \exp\left(i\gamma^n \frac{\partial_l}{\partial\theta^n} \right) \int_0^\infty de_0 \int \frac{dp}{(2\pi)^4} \exp\left\{ ip(x_b - x_a) + i\frac{e_0}{2}\left(p^2 - m^2\right) \right\} \int D\psi \int_0^1 \left(\psi p + m\psi^5\right) d\tau \times \exp\left\{ \int_0^1 \psi_n\dot{\psi}^n d\tau + \psi_n(1)\psi^n(0) \right\}_{\theta=0}. \quad (23)$$

At this stage, in order to be free of the boundary conditions Eq.(15), we replace the integration over odd ψ by one over velocity ω:

$$\psi^n(\tau) = \frac{1}{2}\int \varepsilon(\tau - \tau')\omega^n(\tau')d\tau' + \frac{\theta^n}{2}, \qquad \omega^n(\tau) = \dot{\psi}^n(\tau), \quad \varepsilon(\tau) = sign\,\tau. \quad (24)$$

Obviously the integrations over ω are not restricted, i.e. the boundary conditions are satisfied automatically. Actually, in virtue of the change (24), the boundary term $\psi_n(1)\psi^n(0)$ which presents an ambiguity in performing functional integrations over spinning variables is also eliminated. So, the Green function is written using a condensed notation as:

$$\widetilde{S}_0^c(x_b, x_a) = \frac{1}{2}\exp\left(i\gamma^n \frac{\partial_l}{\partial\theta^n} \right) \int_0^\infty de_0 \int \frac{dp}{(2\pi)^4} \exp\left\{ ip(x_b - x_a) + i\frac{e_0}{2}\left(p^2 - m^2\right) \right\} \int D\omega \int_0^1 \left[p(\varepsilon\omega + \theta) + m\left(\varepsilon\omega^5 + \theta^5\right) \right] d\tau \times \exp\left\{ -\frac{1}{2}\int_0^1 \omega_n \varepsilon\omega^n d\tau \right\}_{\theta=0}. \quad (25)$$

Introducing odd sources $\rho^n(\tau)$ associated to velocities ω^n, we get:

$$\widetilde{S}_0^c(x_b, x_a) = \frac{1}{2}\exp\left(i\gamma^n \frac{\partial_l}{\partial\theta^n}\right)\int_0^\infty de_0 \int \frac{dp}{(2\pi)^4} \exp\{ip(x_b - x_a)$$

$$+ i\frac{e_0}{2}(p^2 - m^2)\Big\}\int_0^1 \left[p\left(\varepsilon\frac{\delta}{\delta\rho} + \theta\right) + m\left(\varepsilon\frac{\delta}{\delta\rho^5} + \theta^5\right)\right] d\tau I(\rho)\Big|_{\theta=0}^{\rho=0}, \quad (26)$$

where $I(\rho)$ is Gaussian path integral over Grassmann variables:

$$I(\rho) = \int D\omega \exp\left\{-\frac{1}{2}\int_0^1 \left[\omega_n \varepsilon\omega^n + \rho_n \omega^n\right] d\tau\right\} \qquad (27)$$

Now, the integration over ω is straightforward and gives

$$I(\rho) = exp\left\{-\frac{1}{2}\int_0^1 \left[\rho_n \varepsilon^{-1}\rho^n\right] d\tau\right\}. \qquad (28)$$

It is easy to verify that the inverse of ε is given by:

$$\varepsilon^{-1}(\tau, \tau') = \frac{1}{2}\frac{\partial}{\partial\tau}\delta(\tau - \tau'). \qquad (29)$$

Now, by using one of the properties of delta functions, Eq.(28) takes the form:

$$I(\rho) = \exp\left\{\frac{1}{4}\int_0^1 \left[\dot{\rho}_n \rho^n\right] d\tau\right\}. \qquad (30)$$

Inserting this result in Eq.(26) and performing differentiations with respect to ρ, we get:

$$\widetilde{S}_0^c(x_b, x_a) = \frac{1}{2}\int_0^\infty de_0 \int \frac{dp}{(2\pi)^4} \exp\left\{ip(x_b - x_a) + i\frac{e_0}{2}(p^2 - m^2)\right\}$$

$$\times\left(p_\mu \frac{\partial_l}{\partial\xi_\mu} + im\gamma^5\right)\exp\left(i\xi_n\gamma^n\right)\Big|_{\xi=0}, \qquad (31)$$

where we have used the easily proved formula:

$$exp\left(i\gamma^n \frac{\partial_l}{\partial\theta^n}\right) f(\theta)\Big|_{\theta=0} = f\left(\frac{\partial_l}{\partial\xi}\right) exp\left(i\xi_n\gamma^n\right)\Big|_{\xi=0}. \qquad (32)$$

Expanding $\exp(i\xi_n\gamma^n)$ to the first order only (the other terms do not contribute) and next performing differentiation with respect to ξ, we get:

$$\widetilde{S}^c(x_b, x_a) = \frac{i}{2}\int_0^\infty de_0 \int \frac{dp}{(2\pi)^4}(p_\mu\gamma^\mu + m\gamma^5)$$

$$\times \exp\left\{ip(x_b - x_a) + i\frac{e_0}{2}(p^2 - m^2)\right\}. \qquad (33)$$

Finally, integrating over the proper time e_0 and multiplying the result by γ^5 taking into account:

$$\widetilde{\gamma}^\mu = \gamma^5\gamma^\mu, \quad \widetilde{S}_0^c = S_0\gamma^5, \quad \left(\gamma^5\right) = -1, \qquad (34)$$

we obtain the familiar expression of the free propagator of a spinning particle:

$$S_0^c(x_b, x_a) = -\int \frac{dp}{(2\pi)^4} \exp\{-ip(x_b - x_a)\}\frac{p\gamma + m}{p^2 - m^2 + i\varepsilon}. \qquad (35)$$

CONCLUSION

We have succeeded in calculating within the framework of path integrals the Green's function for free spinning particles using a general representation for the propagator via bosonic and fermionic path integrals. The procedure of integrating out the fermionic variables is based on a convenient transformation, which allows us to be free of the boundary conditions and by the same time to get a Gaussian path integral easily performed. Thanks to the introduction of Grassmannian sources, the integration over velocities is readily carried out. We can say that this elementary problem is a good example to present some useful techniques which can be used to solve more complicated problems such as plane wave [6], constant electromagnetic field [7], where we are faced by non trivial path integral.

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