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BASIC PROPERTIES OF *I*-ADIC ALGEBRA

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Abstract

In this paper we study the basic properties of the *I*-adic algebras (spectrum, radius of regularity, group of inversible elements, entire functions, etc.).

***Key words**: Algebra, Topology, I-adic, spectrum, radius of regularity, entire functions.*

Résumé

Dans ce papier, nous étudions les propriétés de base des algèbres *I*-adiques (spectre, rayon de régularité, groupe des éléments inversibles, fonctions entières, etc.).

***Mots clés**: Algèbre, Topologie, I-adique, spectre, rayon de régularité, fonctions entières.*

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ملخص

في هذا المقال قمنا بدراسة الخواص الأساسية للجبر *I*-adic (طيف، قطر التنظيم، زمرة العناصر القابلة للقلب، الدوال الطبيعية).

***الكلمات المفتاحية:** جبر، تبولوجيا، I-adic ، طيف، قطر التنظيم ، الدوال الطبيعية.*

An I-adic **C**-algebra is an algebra endowed with a topology called I-adic, denoted τ_I and defined by the sequence $(I^n)_{n\geq 0}$, of ideals, as a fundamental system of neighborhoods of zero. Notice that, if **C** is endowed with the usual absolute value, then this algebra is not topological. But, it is so when **C** is endowed with the discrete topology. We examine the compatibility of the *I*-adic topology with the usual operations. The finite product, the quotient, the completion and the unitization behave well. But, this is not the case for the infinite product, the projective limit, the inductive limit and the induced topology. We consider topologies which play analogous roles of these. We give necessary and sufficient conditions for the group $G(A)$, of invertible elements, which is always a topological group, to be open. In this case, it is also closed. If we consider $Sp_A(a)$ as a subset of A, then $\overline{Sp_A(a)} = Sp_A(a) \bigcap_{n\geq 0} I^n$. We also prove that $Sp_A(a) = \bigcup_{M\in\Gamma} Sp_{A_M}(a)$, where A_M is the localization of A on M and Γ is the family of all maximal ideals of A. We also obtain that $r(a) < 1$, implies that $(e-a)^{-1} = e + \sum_{n\geq 1} a^n$ where r is the radius of regularity. In general, $r(a)$ is different from the spectral radius $\sup_{\lambda\in Sp_A(a)} |\lambda|$. We notice that if I is not in the Jacobson radical $R(A)$, of A, then τ_I can not be complete. The analogous of Johnson's results in the Banach case, is that the discrete topology is the unique complete *I*-adic topology on semi-simple algebra. The Gelfand transformation is continuous if, and only if, every multiplicative linear functional is continuous. We prove that an entire series operate on an element a in (A,τ_I) if, and only if, a is the radical "$\sqrt{I}$" of I.

Throughout this paper, A will denote a commutative **C**-algebra with unit e and I an ideal of A. The I-adic topology on A, denoted by τ_I, is the topology defined by the sequence $(I^n)_{n\geq 0}$ of ideals, as a system of neighborhoods of zero. If the complex field **C** is endowed with the discrete topology, then (A,τ_I) is a topological algebra with (globally) continuous product.

1. STABILITY PROPERTIES

The I-adic topology behave well with some standard constructions but not with others. For the latters, we give situations where it does or provide a substitute one.

The next example shows that the restriction of τ_I to a subalgebra B of A is not, in general, I-adic.

Example 1.1. Denote by $\mathbf{C}[X,Y]$ the algebra of complex polynomials of two indeterminates. Endow it with the I-adic topology given by $I = X\mathbf{C}[X,Y]$. Consider the subalgebra $B = Y\mathbf{C}[X,Y]$ of $\mathbf{C}[X,Y]$. A system of neighborhoods of 0 for the topology $(\tau_I)/B$ is $(X^n B)_{n\geq 0}$. Let J be an ideal of B such that $(\tau_I)/B = \tau_J$. Then, there exists a positive integer n such that J^n is contained in XB, hence $(\tau_I)/B = \tau_{XB}$, which is a contradiction because τ_{XB} is strictly finer than $(\tau_I)/B$.

However, there exists in general an I-adic topology which is the coarset I-adic topology on B making the canonical injection continuous. This is the topology $\tau_{I\cap B}$.

A finite product of I-adic algebras is I-adic. But this is not the case, in general, for an infinite product as the following example shows.

Example 1.2. Let $(A_l, \tau_{I_l})_l$ be an infinite family of Hausdorff I-adic algebras. Then, τ_J is not coarset than the product topology, for every ideal $J = \prod_l J_l$ of $\prod_l A_l$, with $J_l \neq A_l$ except for a finite number of indexes. If $\prod_l I_l^{n_l}$ is contained in $\prod_l J_l$, with $n_l = 0$ except for a finite number for indexes, then J_l contains $I_l^{n_l}$, for every l. So for $n_l = 0$, $J_l = A_l$, a contradiction. Now, if $J = \prod_l J_l$ with $J_l = A_l$ except for a finite number of indexes, then for $J_l = A_l$, π_l is not continuous.

Here is an example where there exists an I-adic topology on an infinite product of I-adic algebras. It is the coarset of I-adic topologies such that the canonical projections are continuous. Let $(A_l, \tau_{P_l})_l$ be an infinite family of I-adic algebras, where P_l is a prime ideal of A_l, for every l. Consider τ_I, where $I = \prod_l P_l$.

A quotient of an I-adic algebra is always I-adic, and if J is an ideal of A, then the topology $\tau_{\frac{I+J}{J}}$ defined on $\frac{A}{J}$ by $\frac{(I+J)}{J}$, coincides with the quotient topology.

Concerning completion and proceeding as in ref. [2], we obtain the following result.

Theorem 1.3. *Let (A, τ_I) be a Hausdorff I-adic algebra. Then, there exist a complete I-adic algebra denoted $(\hat{A}, \tau_{\hat{I}})$ and a continuous monomorphism $\Phi : (A, \tau_I) \to (\hat{A}, \tau_{\hat{I}})$ such that $\Phi(A)$ is dense subalgebra of $(\hat{A}, \tau_{\hat{I}})$.*

The next example shows that the projective limit of I-adic topologies is not, in general, I-adic.

Example 1.4. Consider the projective system $((A_\lambda, \tau_\lambda), f_{\lambda\mu})_{\lambda,\mu\in\mathbf{C}}$, where $A_\lambda = \mathbf{C}[X]$, $\tau_\lambda = \tau_{(X+\lambda)A_\lambda}$ for every λ in $\mathbf{C}$. Then $\varprojlim (A_\lambda, f_{\lambda\mu})_{\lambda,\mu\in\mathbf{C}} = \prod_{\lambda\in\mathbf{C}} A_\lambda$ and the projective topology is the product one which is not I-adic by (Example 1.2.). However, there is on $\prod_{\lambda\in\mathbf{C}} A_\lambda$ an I-adic topology which is the coarset I-adic topology such that the canonical projections are continuous. It is given by the ideal $\prod_\lambda (X+\lambda)A_\lambda$. This is a particular situation of the following.

Proposition 1.5. *Let $((A_l, \tau_{P_l}), f_{kl})_{k,l\in L}$, be a projective system of I-adic algebras, where P_l is a prime ideal of A_l, for every l in L. Put $A = \varprojlim (A_l, f_{kl})_{k,l\in\Gamma}$. The I-adic topology τ_I defined by $I = \varprojlim (P_l, f_{kl})_{k,l\in L}$ is the coarset I-adic topology, on A, such that the projections $\pi_l : (A, \tau_I) \to (A_l, \tau_{P_l})$ are continuous.*

In general, an inductive limit of I-adic topologies is not always I-adic. Actually, there is no a finest I-adic topology on an inductive system of I-adic algebras such that the canonical injections are continuous.

Proposition 1.6. *Let $((A_n, \tau_{I_n}), f_{nm})_{n\geq 0}$ be an inductive system of Hausdorff I-adic algebras and $A = \varinjlim (A_n, f_{nm})_{n\geq 0}$. Then, for every ideal I of A such that the injection $i_n : (A_n, \tau_{I_n}) \to (A, \tau_I)$ are continuous, there exists an ideal J of A that the injection $i_n : (A_n, \tau_{I_n}) \to (A, \tau_J)$ are continuous and τ_J is strictly finer than τ_I.*

Proof. Since $i_n : (A_n, \tau_{I_n}) \to (A, \tau_I)$ is continuous, there exists, for every integer $n \geq 0$, an integer $r_n > 0$ such that $i_n(I_n^{r_n})$ is contained in I, so I contains $\bigoplus_{n\geq 0} i_n(I_n^{r_n})$. It suffices to put $J = \bigoplus_{n\geq 0} i_n(I_n^{r_n})$.

2. THE GROUP OF INVERTIBLE ELEMENTS AND JACOBSON'S RADICAL

We endow $G(A)$ with the topology induced by τ_I. The map $a \mapsto a^{-1}$ is a homeomorphism of the topological group $G(A)$ onto itself. As a different fact, with the Banach case, the openess of $G(A)$ is in relation with the radical.

Proposition 2.1. *The following assertions are equivalent.*

(i) *$G(A)$ is open.*
(ii) *$I \subset RadA$.*
(iii) *Every maximal ideal of A is closed for τ_I.*

Notice that if *RadA* contains *I*, then *G*(*A*) is also closed for τ_I.

The spectrum of an element *a* of *A*, denoted $Sp_A(a)$ is $\{\lambda\in\mathbf{C}/a-\lambda e\notin G(A)\}$. As a subset of **C**, $Sp_A(a)$ is closed. If we consider $Sp_A(a)$ as a subset of *A*, then we have the following, where $\overline{Sp_A(a)}$ denotes the closure of $Sp_A(a)$ in (A,τ_I); whence de closedness of $Sp_A(a)$ if (A,τ_I) is Hausdorff.

Proposition 2.2. *Let* (A,τ_I) *be an I-adic algebra. Then* $\overline{Sp_A(a)}=Sp_A(a)+\bigcap_{n\geq 0}I^n$*, for every element a in A.*

Proof. Let *y* be an element of $\overline{\left(Sp_A(a)+\bigcap_{n\geq 0}I^n\right)}$, we have $y=\lambda+b$, with λ is in $Sp_A(a)$ and *b* in $\bigcap_{n\geq 0}I^n$. Hence $y=\lambda-b$ is in $(y+I^n)\bigcap Sp_A(a)$, i.e., *y* is in $\overline{Sp_A(a)}$. Conversely, let *y* be an element of $\overline{Sp_A(a)}$ and n_0 a positive integer.

Then $(y+I^{n_0})\bigcap Sp_A(a)$ is not void, so there exist an element λ_0 in $Sp_A(a)$ and an element b_0 in I^{n_0} such that $y=\lambda_0+b_0$. There exist also an element λ_i in $Sp_A(a)$ and an element b_i in I^{n_i} such that $y=\lambda_i+b_i$. Thus $\lambda_0-\lambda_i=b_i-b_0$. If $\lambda_0\neq\lambda_i$, then b_i-b_0 is invertible which is contradictory. Hence $b_i=b_0$ and *y* is in $Sp_A(\mathrm{a})+\bigcap_{n\geq 0}I^n$.

Proposition 2.3. *Let A be an algebra, then for every element a in A,* $Sp_A(a)=\bigcup_{M\in\Gamma}Sp_{A_M}\left(\frac{a}{e}\right)$*, where* Γ *is the set of all maximal ideals of A.*

Proof. Let λ be an element of $Sp_{A_M}\left(\frac{a}{e}\right)$, then $\lambda e-\frac{a}{e}$ is not invertible in A_M and so is $\lambda e-a$ in *A*. Hence λ is in $Sp_A(a)$. For the converse, if λ is an element of $Sp_A(a)$, then $\lambda e-a$ is not invertible in *A* and so it is an element of a maximal ideal *M* of *A*. Hence $\lambda e-\frac{a}{e}$ is in MA_M. Thus $\lambda e-\frac{a}{e}$ is not invertible in A_M, i.e., λ is in $Sp_{A_M}\left(\frac{a}{e}\right)$.

Recall that an algebra is said to be semi-local if it has a finite number of maximal ideals. By the previous proposition, the spectrum of every element *a* in a semi-local algebra *A,* is void or finite.

We suppose that (A,τ_I) is a Hausdorff *I*-adic algebra. The topology can be defined by the metric *d*. The radius of regularity of an element *a*, denoted *r*(*a*), is defined by $r(a)=\underset{n\geq 1}{Inf}(d(a^n,0))^{1/n}$. One easily checks that $r(a)=\lim_n(d(a^n,0))^{1/n}$.

Remark 2.4. In general, the quantity *r*(*a*) is different from the spectral radius $\underset{\lambda\in Sp_A(a)}{Sup}|\lambda|$. Indeed, consider the algebra $A=\mathbf{C}[[X]]$ of formal power series. Endow it with the *I*-adic topology defined by $I=X\mathbf{C}[[X]]$. For $f(X)=3+X$, $r(f(X))=1$ because $(f(X))^n$ is in $I^0\backslash I$, for every integer $n\geq 0$. But $\underset{\lambda\in Sp_A(a)}{Sup}|\lambda|=3$ because $Sp_A(f(X))=\{3\}$.

By the same argument as in the proof of theorem 9. p. 12 of ref. [1], we obtain the following results.

Proposition 2.5. *Let* (A,τ_I) *be a complete I-adic algebra and a an element of A such that* $r(a)<1$*. Then* $(e-a)$ *is invertible and* $(e-a)^{-1}=e+\sum_{n\geq 1}a^n$.

Corollary 2.5. *Let* (A,τ_I) *be a complete I-adic algebra with unit e. Then, each element a of A with* $d(e-a,0)<1$*, is invertible.*

As a consequence, we have the following result.

Proposition 2.6. *Let* (A,τ_I) *be a complete I-adic algebra. Then G*(A) *is open; hence* (A,τ_I) *is never semi simple.*

Proof. Let *a* be an element of *G*(A).Then *G*(A) contains $(a+I)$. Therefore, by proposition 2.5, $(e+a^{-1}b)$ is invertible, and *G*(A) being a group, $(a+b)=a(e+a^{-1}b)$ is also in *G*(A). Then see proposition 2.1.

We, of course, do have different *I*-adic topologies on a given algebra *A*. First notice that, if *I* and *J* are two ideals of *A,* then τ_I is finer than τ_J if, and only if, there exists an integer $n\geq 0$ such that *J* contains I^n. Now, consider the algebra *C*(*X*) of all continuous, complex valued functions on *X*, where *X* is a Haussdorff completely regular topological space. Let *x* and *y* be two different elements of *X*. Then, by corollary 2.2 of ref.[3], $I_x=\{f\in C(X)/f(x)=0\}$ and $I_y=\{f\in C(X)/f(y)=0\}$ are two different maximal ideals of *C*(*X*). Hence, τ_{I_x} and τ_{I_y} are incomparable. The analogous of Johnson's result for a semi-simple Banach algebra is the following: In a semi simple algebra *A*, the discrete topology is the unique complete *I*-adic topology on *A*. We now give a result where semi-simplicity is not involved.

Proposition 2.7. *There exists on an algebra A a unique I-adic topology which is not discrete and not trivial if, and only if:*

i) A admits a unique prime ideal P.

ii) Each ideal of A contains a power of P.

Proof. If *P* and *P'* are two prime ideals of *A,* then by hypothesis, $\tau_P=\tau_{P'}$; so there exist two positive integers *n, m* such that *P'* contains P^n and *P* contains $(P')^m$. Hence *P'* contains *P* and *P* contains *P'*, i.e., *P* = *P'*. On the other hand, since for any ideal *I* of *A* $\tau_I=\tau_P$, there exists a positive integer *n* such that *I* contains P^n. Whence necessity. For the converse, if *I* is an ideal of *A*, then *P* contains *I* and so τ_I is finer than τ_P. Since there exists a positive integer n such that I contains P^n, we have τ_P finer than τ_I.

The last proposition applies, in particular, to algebras of valuation in which the non zero ideals are the sets P^n, $n_0 \geq 0$, with P the unique maximal ideal.

3. GELFAND TRANSFORMATION

Concerning the continuity of mutiplicative linear functionals, we have the following.

Proposition 3.1. *Let (A,τ_I) be an I-adic algebra and χ a multiplicative linear functional. Then χ is continuous if, and only if, $Ker\chi$ contains I.*

Proof. Suppose that $Ker\chi$ contains I. Let x_0 be in A and $\varepsilon>0$. Then $B(\chi(x_0),\varepsilon)$ contains $\chi(x_0+I)$. Conversely, suppose that I is not contained in $Ker\chi$. Then, by the maximality of $Ker\chi$, I^n is not contained in $Ker\chi$, for any integer $n\geq 0$. So $I^n+Ker\chi=A$. Since χ is continuous, there exists, for $\varepsilon>0$, an integer $n\geq 0$ such that $B(0,\varepsilon)$ contains $\chi(I^n)$. Hence $\mathbf{C}=\chi(A)=\chi(I^n+Kerf\chi)=\chi(I^n)$ is contained in $B(0,\varepsilon)$ which is absurd.

Corollary 3.2. *Let (A,τ_I) be an I-adic algebra such that G(A) is open. Then every multiplicative linear functional is continuous.*

We consider the Gelfand transformation $x\mapsto\hat{x}$. We denote by $m^{\#}(A)$ (resp. $m(A)$) the set of all multiplicative (resp. continuous multiplicative) linear functionals on A. Suppose $m^{\#}(A)\neq\{0\}$ and endow $C(m^{\#}(A),\mathbf{C})$ with the topology defined by the sets $V(f,\chi_1,\ldots,\chi_n,\varepsilon)=\{g\in m^{\#}(A)/|f(\chi_i)-g(\chi_i)|<\varepsilon,\ 1\leq i\leq n\}$ with $\varepsilon>0$, $\chi_1,\ldots,\chi_n$ in $m^{\#}(A)$; as a system of neighborhoods of an element f in $m^{\#}(A)$. Here is an example where $m^{\#}(A)\neq m(A)$.

Example 3.3. Let $(A,\|.\|)$ be a commutative Banach algebra such that $Rad(A)$ contains a non nilpotent element x. In ref. [4], S. Rolewicz has constructed a sequence $(a_{k,n})_{k,n}$ such that, for every k, m, n in $\mathbf{N}$, $a_{k,n}\geq 1$ and $a_{k,n+m}\leq a_{k+1,n}a_{k+1,m}$; and he has considered $B=\left\{(x_n)_{n\geq 1}\subset A: \sum_{n\geq 1} a_{k,n}\|x_n\|<+\infty,\ \forall k\right\}$. Endowed with the usual operations, the convolution product and the norm defined by $\|(x_n)_n\|=\sum_{n\geq 1}\|x_n\|$, B is a normed algebra containing A and verifying $Rad(B)\neq\bigcap_{\chi\in m^{\#}}Ker\chi$. This algebra admits a maximal ideal M of infinite codimension. If we endow it with the I-adic topology defined by M, then there is no continuous multiplicative linear functional on B.

Proposition 3.4. *The Gelfand transformation $\Lambda:(A,\tau_I)\to C(m^{\#}(A),\mathbf{C})$, $x\mapsto\hat{x}$, is continuous if, and only if, $m^{\#}(A)=m(A)$.*

Proof. By the previous proposition, if χ is continuous, then $Ker\chi$ contains I. So $V(\hat{x},\chi,\varepsilon)$ contains $\Lambda(x+I)$, for every $\varepsilon>0$. Conversely, suppose that there exists a non continuous multiplicative linear functional χ on A. Then, I is not contained in $Ker\chi$ and so is for I^n, $n\geq 0$. Hence, for every integer $n\geq 0$, $I^n+Ker\chi=A$. For $\varepsilon>0$, there exists an integer $n_0\geq 0$ such that $V(0,\chi,\varepsilon)$ contain $\Lambda(I^{n_0})$. But $I=I^{n_0}+Ker\chi\bigcap I$. So $\chi(I)$ is contained in $B(0,\varepsilon)$. Hence $\chi(I)=0$; a contradiction.

4. ENTIRE FUNCTIONS

Now, we examine the behavior of entire functions. They not operate on the whole algebra.

Proposition 4.1. *Let (A,τ_I) be an I-adic algebra and $f(z)=\sum_n\lambda_n z^n$ an entire function, which is not a polynomial. The series $\sum_n\lambda_n a^n$ converges in A if, and only if, a is in $\sqrt{I}$.*

Proof. Sufficiency: Let n_0 be the smallest positive integer such that a^{n_0} is in I. For every positive integers $r>0$, $s>0$ and $m\geq n_0 s$, we have

$$\sum_{0\leq n\leq m+r}\lambda_n a^n-\sum_{0\leq n\leq m}\lambda_n a^n=\sum_{m+1\leq n\leq m+r}\lambda_n a^n=a^{n_0 s}\left(\sum_{m+1\leq n\leq m+r}\lambda_n a^{n-n_0}\right),$$

which is in I^s. Hence, $\sum_n\lambda_n a^n$ is Cauchy in (A,τ_I), thus it is convergent. Necessity: if $\sum_n\lambda_n a^n$ converges in (A,τ_I), then there exists an integer n_0 such that, for every integers $r>0$, $m\geq n_0$, we have $\sum_{0\leq n\leq m+r}\lambda_n a^n-\sum_{0\leq n\leq m}\lambda_n a^n\in I$. Consider an integer $m\geq n_0$ such that $\lambda_{m_0+1}\neq 0$. Since $\sum_{0\leq n\leq m_0+1}\lambda_n a^n-\sum_{0\leq n\leq m_0}\lambda_n a^n$ is in I, then $\lambda_{m_0+1}a^{m_0+1}$ is in I and it follows that a^{m_0+1} is in I.

Example 4.2. Consider the algebra $A=\mathbf{C}[[X]]$ of formal power series. Endow it with the I-adic topology defined by $I=\mathbf{C}[[X]]$. By theorem 4.1, a non polynomial series $\sum_{n\geq 0}\lambda_n a^n$ on A converges if, and only if, a is in I.

5. ILLUSTRATION

To illustrate phenomena we encountered we, at last, examine them on a very classical example.

Example 5.1. Let X be a completely regular Hausdorff space. Denote by $C(X)$ the semi-simple algebra of all continuous, complex valued functions on X with the usual pointwise operations. The closed maximal ideals in $C(X)$ are the subsets $I_x=\{f\in\mathbf{C}(X)/f(x)=0\}$, with $x\in X$ ([3]). We examine the algebra $C(X)$ endowed with the I_x-adic topology $\tau_x, x\in X$. Since X is completely regular Hausdorff space, $I_x\neq C(X)$ and $I_x^n\neq\{0\}$, for every integer $n\geq 0$. Hence, τ_x is not trivial nor discrete. Let $x\neq y$ be two different elements in X. Then, τ_x and τ_y are incomparable. The topology τ_x is not Hausdorff. Indeed, let $y\neq x$ in X. Then, there exists a continuous function $f:X\to[0,1]$, such that $f(x)=0$ and $f(y)=1$. So, $f\in\bigcap_{n\geq 0}I_x^n$, since for every integer $n\geq 1$, $\sqrt[n]{f}\in I_x$. The group $G(C(X))$ of invertible elements in $C(X)$ is the set $\{f\in C(X)/f(y)\neq 0,\ \forall y\in X\}$. It is not open. Indeed, if there exists n in $\mathbf{N}$ such that $(1+I_{x_0}^n)$ is contained in $G(C(X))$, then, for every y in X and every f in I_x, $f^n(y)\neq 1$ which is not true, because if $y\neq x$ in X, then there exists f in $C(X)$ such that $f(x)=0$ and $f(y)=1$. So, $f^n(y)=f(y)\ldots f(y)=1$. As it is known, the nonzero multiplicative linear functionals on $C(X)$ are in one to one correspondence with the points of X via the relation $y\mapsto\chi_y$, where $\chi_y(f)=f(y)$, for $f\in C(X)$ ([3]). The unique nonzero continuous character on $C(X)$ for τ_x is χ_x. At last, since I_x is a maximal ideal of $C(X)$, a non polynomial series $\sum_{n\geq 0}\lambda_n a^n$ converges if, and only if, a is in I_x.

Example 5.2. Let A_0 be a radical commutative algebra, e.g., $L^1[0,1]$, the multiplication being the convolution operation; $f*g=\int_0^t f(t-s)g(s)ds$. A proper ideal I of the unitization A of A_0 is of the form $I_0\times\{0\}$, where $I_0\neq\{0\}$ is an ideal of A_0. The Jacobson radical of A is A_0 and the group of invertible elements of A is $G(A)=\{a_0+\lambda/a_0\in A_0,\ \lambda\in\mathbf{C}^*\}$. It is open, for every I-adic topology on A (cf. Proposition 2.1). Next, we remark that the unique nonzero character χ on A is defined as follows $\chi:A\to\mathbf{C},\ a_0+\lambda\to\lambda$. By the theorem 3.1, it is continuous, for every topology on A.

Example 5.3. Let $C(X)$ be the semi-simple algebra of all continuous complex valued functions on X, where X is a completely regular Hausdorff space. And let A be the unitization of a given radical algebra A_0. Then, the algebra $B=C(X)\times A$ is not radical and not semi-simple. The group of invertible element in B is $\{f\in C(X)/f(y)\neq 0,\ \forall y\in X\}\times\{a_0+\lambda/a_0\in A_0,\lambda\in\mathbf{C}^*\}$. Consider on B the I-adic topology defined by the ideal $I=I_x\times J$, where $I_x=\{f\in C(X)/f(x)=0\}$ and J an ideal of A. The group $G(B)$ is open if, and only if, $x=0$ (cf. Proposition 3.2). Let χ be a character on B. Then, $\chi\circ i_{C(X)}$ (resp. $\chi\circ i_A$) is a character on $C(X)$ (resp. on A), where $i_{C(X)}:C(X)\to B,\ f\mapsto(f,0)$ and $i_A:A\to B$, $a\mapsto(a,0)$. Hence, the nonzero multiplicative linear functional on B are in one to one correspondence with the points of X via the relation $y\mapsto\chi_y$, where $\chi_y(f,a_0+\lambda)=f(y)+\lambda$. A character χ_y is continuous if, and only if, $y=x$.

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