ON MATRICES ARISING IN RETARDED DELAY DIFFERENTIAL SYSTEMS

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Résumé

Dans cet article, on considère une classe de système différentiels retardés et à laquelle on associe une matrice système sur $R[s,z]$, l'anneau des polynômes à deux indéterminés s et z. Ensuite, en utilisant la notion de la matrice forme de Smith sur $R[s,z]$, on étend un résultat de caractérisation obtenu précédemment [5] sur les formes canoniques, à un cas plus général.

***Mots clés**: Systèmes linéaires, systèmes différentiels retardés, contrôlabilité, matrices systèmes, formes canoniques.*

Abstract

In this paper, we consider a class of differential systems known as retarded delay differential systems and to which we associate a system matrix over $R[s,z]$, the ring of polynomials of two indeterminates s and z. Then using the notion of Smith form matrix over $R[s,z]$, we extend a previously obtained characterization results [5], on canonical forms, to a more general case.

***Key words**: Linear systems, retarded delay differential systems, controllability, system matrices, canonical forms.*

AMS Classification: 93B

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ملخص

في هذا البحث، نعتبر فئة جمل تفاضلية تباطئية ونرفق بها مصفوفة جملة على $R[s,z]$، حلقة كثيرا الحدود ذات متغيرين s و z . بعد ذلك، باستعمال مفهوم مصفوفة شكل سميث على $R[s,z]$ ، تمدد نتيجة معيارية محصل عليها سابقا [5] ، على الأشكال القانونية، إلى حالة أكثر عموما.

***الكلمات المفتاحية**: جمل خطية، جمل تفاضلية تباطئية، تحكم، مصفوفات جمل، أشكال قانونية.*

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1. PRELIMINARIES

One of the important motivation behind this research work concerning canonical form from system matrices (Smith form & companion form) associated to "RDDS" is that the crucial role that may be played by this canonical forms in unifying the research work in the theory of linear systems. They also play a fundamental role in the study of structural properties of systems as controllability, observability and minimality.

Now, we give the following result which can be considered as a generalization of a result given in Barnett [1] to the case of n^{th} order L.D.E. of variable coefficients of the form

$$z^{(n)}+k_1(t)z^{(n-1)}+...+k_{n-1}(t)z'+k_n(t)z=\beta(t)u(t) \tag{1}$$

which can be transformed into the form

$$w'=C(t)w+d(t)u(t) \tag{2}$$

where the companion matrix $C(t)$ associated with (1) is

$$C(t)=\begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots \\ \vdots & \vdots & \vdots & \dots & 1 \\ -k_n(t) & -k_{n-1}(t) & -k_{n-2}(t) & \dots & -k_1(t) \end{bmatrix},\ d(t)=\begin{bmatrix} 0 \\ 0 \\ \vdots \\ \vdots \\ \beta(t) \end{bmatrix}$$

and $w=\left[z\ z'\ z''\dots z^{(n-1)}\right]^T$, where the solution of (1) is by definition the first component of w i.e. z and not the whole vector w.

2. MATRICES ARISING IN RETARDED DELAY DIFFERENTIAL SYSTEMS

Now, we try to get an analogous result for systems described by retarded delay differential equations where the role of the matrix $A(t)$ in thermodynamical system $x'(t)=A(t)x(t)+B(t)u(t)$ is played by the matrix $A(s,z)=sI_n-A(z)$ (which can be considered as an extension of the O.D.E.

or D.E.). Let the system of retarded delay differential equation given by

$$\begin{cases} x(t)-\sum_{i=1}^{r} A_i x(t-ih)=\sum_{j=1}^{s} B_j u(t-jh) \\ y(t)=\sum_{k=1}^{q} C_k x(t-kh) \end{cases} \tag{3}$$

where $x(t)$ is an n-column state vector, $u(t)$ an m-column control vector, and $y(t)$ is an p-column out put vector, h is a positive constant and A_i, B_j, C_k, $1\le i\le r$, $1\le j\le s$, $1\le k\le q$, are $n\times m$, and $p\times n$ constant matrices respectively. By taking Laplace transform of (3) and assuming zeros initial conditions we get the following matrix

$$M(s,z)=\begin{bmatrix} A(s,z) & B(z) \\ -C(z) & 0 \end{bmatrix} \tag{4}$$

with $A(s,z)=sI_n-A(z)$, where s and z stand for differential and delay operators respectively.

Since the results to follow depend strongly on the idea of Smith form of a matrix over $R[s,z]$, so we give the following definition which is an extension of the 1-D case given in [3].

Definition 1. *Let $A(s,z)$ be a polynomial matrix over the ring $R(s,z)$, we define the Smith form $S(s,z)$ of $A(s,z)$ by*

$$S(s,z)=\begin{cases} \begin{bmatrix} R(s,z) \\ 0 \end{bmatrix} & \text{if } p\ge q \\ R(s,z) & \text{if } p=q \\ \text{or } \begin{bmatrix} R(s,z) & 0 \end{bmatrix} & \text{if } p<q \end{cases} \tag{5}$$

where $R(s,z)=diag\{i_1(s,z), i_2(s,z),...,i_m(s,z)\}$, *with* m = min(p,q). *The elements $i_k(s,z)$ are given by*

$$i_k(s,z)=\begin{cases} \frac{d_k(s,z)}{d_{k-1}(s,z)}, & k=1,2,...,r \\ 0, & k=r+1,...,m \end{cases} \tag{6}$$

where $r=rank\,A(s,z)$, $d_0(s,z)=1$ *and* $d_k(s,z)$ *is defined as the greatest common divisor of all the k^{th} order minors of $A(s,z)$.*

Now, we give the following rank conditions on the matrices arising from retarded delay differential systems.

Theorem 2. *The matrix* $A(s,z)=sI_n-A(z)$ *over* $R[s,z]$ *is equivalent to the Smith form*

$$S(s,z)=\begin{bmatrix} I_{n-1} & 0 \\ 0 & |sI_n-A(z)| \end{bmatrix} \tag{7}$$

if there exists a column n-vectors $B(z)$ over $R[z]$ such that $rank[sI_n-A(z) \quad B(z)]=n, \forall(s,z)\in C^2$.

(Note the analogy with the previous results). We also note that the condition (8) is equivalent to the condition

$$rank\left[B(z) \quad A(z)B(z) \quad ... \quad A^{n-1}(z)B(z)\right]=n, \forall z\in C \tag{9}$$

(i.e. $R^n[z]$ controllability).

For a proof of the above result, see [2]. For more related results on system matrices, see [4].

Now, we try to get a companion form matrix representation for the matrix $A(s,z)=sI_n-A(z)$.

Theorem 3. *Suppose that the condition (9) is satisfied, then the matrix* $A(s,z)=sI_n-A(z)$ *over* $R[s,z]$ *can be transformed by an equivalence transformation into the canonical form (companion form)*

$$\widetilde{A}(s,z)=sI^n-C(z) \tag{10}$$

where $C(z)$ is the companion matrix over $R[z]$ given by

$$C(z)=\begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & 1 \\ a_n(z) & a_{n-1}(z) & \cdots & \cdots & a_1(z) \end{bmatrix} \tag{11}$$

where $a_k(z)$ are the coefficients in the characteristic polynomial of $A(z)$ i.e. $|sI_n\text{-}A(z)|=\sum_{k=1}^{n}-a_k(z)s^{n-k}$, $a_0(z)=1$.

Proof. Using a characterization result given in [2] and that the fact

$$\left[|sI_n-A(z)| \quad \vdots \quad E_n\right]$$

has no zeros, we get the matrix $\widetilde{A}(s,z)=sI_n-C(z)$, is equivalent to the Smith form

$$S(s,z)=\begin{bmatrix} I_{n-1} & 0 \\ 0 & |sI_n-A(z)| \end{bmatrix}$$

and since $|sI_n-C(z)|=|sI_n-A(z)|$, then $A(s,z)=sI_n-A(z)$ is equivalent over $R[s,z]$ to $\widetilde{A}(s,z)=sI_n-C(z)$. ■

Now, we give a result concerning matrices in the state space form $A(s,z)=sI_n-A(z)$, and which can be considerate as an extension of a result given in [5].

Theorem 4. *The matrix* $A(s,z)=sI_n-A(z)$ *is equivalent to the Smith form*

$$S(s,z)=\begin{bmatrix} I_{n-k} & 0 & 0 & \dots & 0 \\ 0 & d_{n-(k-1)}(s,z) & 0 & \dots & 0 \\ 0 & 0 & d_{n-(k-2)}(s,z)\dots & & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & & \dots d_n(s,z) \end{bmatrix}, k\in N^* \tag{12}$$

if the matrix $A(z)$ is similar over $R[z]$ to the block companion matrix

$$C(z)=\begin{bmatrix} C_1(z) & 0 & 0 & \cdots & 0 \\ 0 & C_2(z) & 0 & \dots & 0 \\ 0 & 0 & C_3(z) & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \dots & C_k(z) \end{bmatrix} \tag{13}$$

where $C_i(z)$ are $n_i\times n_i$ $(i=\overline{1,k})$ companion matrices having characteristic polynomials $d_{n-(k-i)}(s,z)$, $(i=\overline{1,k})$ i.e.

$\left|sI_n - C(z)\right| = d_{n-(k--i)}(s,z)$, *where* n_i $(i = 1,k)$ *are given by the degrees in s of* $d_{n-(k--i)}(s,z)$, $(i=\overline{1,k})$.

Proof. To show the necessity, we suppose that $A(s,z)=sI_n-A(z)$ is equivalent over $R[s,z]$ to $S(s,z)$ in (12). Then, by elementary rows and columns operations on $S(s,z)$, we get $S(s,z)$ equivalent to

$$S_k(s,z)=\begin{bmatrix} I_{n_1-1} & 0 & 0 & 0 & \dots & \dots & 0 & 0 \\ 0 & I_{n-(k-1)} & 0 & 0 & \dots & \dots & 0 & 0 \\ 0 & 0 & I_{n_2-1} & 0 & \dots & \dots & 0 & 0 \\ 0 & 0 & 0 & I_{n-(k-2)} & \dots & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & I_{n_3-1} & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & I_{n_k-1} & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & d_n \end{bmatrix} \tag{14}$$

where $d_{n-(k-i)}$ denote $d_{n-(k-i)}(s,z), 1\le i\le k$. Since these operations on a matrix preserve equivalence, then $A(s,z)$ is equivalent to $S_k(s,z)$ in (14).

And since the matrix $S_k(s,z)$ in (14) is clearly equivalent to the block companion matrix

$$sI_n-C(z)=\begin{bmatrix} sI_{n_1}-C_1(z) & 0 & \dots & 0 \\ 0 & sI_{n_2}-C_2(z) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & sI_{n_k}-C_k(z) \end{bmatrix} \tag{15}$$

where $C_i(z)$ are $n_i \times n_i$ $(i=\overline{1,k})$ are square matrices in companion form respectively, such that

$$det(sI_{n_i}-C_i(z))=d_{n-(k-i)}(z), i=\overline{1,k} \quad (k\ge 2)$$

Hence, the matrix $A(s,z)=sI_n-A(z)$ is equivalent to $sI_n-C(z)$, and since this equivalence transformation (between these system matrices) can be replaced by a similarity transformation, then $A(z)$ is similar to

$$C(z)=\begin{bmatrix} C_1(z) & 0 & \dots & 0 \\ 0 & C_2(z) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & C_k(z) \end{bmatrix} \tag{16}$$

which ends the proof of necessity.

The proof of sufficiency: if we assume that $A(z)$ is similar to $C(z)$ in (16), the $sI_n-A(z)$ is also similar to $sI_n-C(z)$ in (15), and since this last matrix is in companion form and is equivalent to its Smith form

$$S(s,z)=\begin{bmatrix} I_{n-k} & 0 & \dots & 0 \\ 0 & d_{n-(k-1)}(s,z) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & d_n(s,z) \end{bmatrix}$$

then so it is $sI_n-A(z)$, which ends the proof of the theorem.

Now, as an illustrative example: for $n=2k, k\in N^*$, let $S(s,z)$ be the matrix of the form

$$S(s,z)=\begin{bmatrix} I_{2k-k} & 0 & \dots & 0 \\ 0 & d_{2k-(k-1)}(s,z) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & d_{2k}(s,z) \end{bmatrix} \tag{17}$$

By (k-1) operations on the rows (respectively columns) of the matrix $S(s,z)$ in (17), we obtain

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & d_{2k-(k-1)} & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & 0 & d_{2k-(k-2)} & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \dots & d_{2k-1} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & d_{2k} \end{bmatrix}$$

where d_{2k-i} denote $d_{2k-i}(s,z), i=\overline{0,k-1}$, and in the above case we have $I_{n_1\text{-}1}=[1]; I_{n_2\text{-}1}=[1]; I_{n_k\text{-}1}=[1]$. i.e. I_{n_i} $(i=\overline{1,k})$ are all matrices of one element equals 1.

We note that above results is a generalization of a previous result given in [5] for the case of $k = 3$.

CONCLUSION

In this paper, a companion form for a matrix of the form $A(s,z)=sI_n-A(z)$, which arises in the study of e.g. retarded delay differential equations was presented, and a characterization result concerning the matrices was extended to a more general case. A similar study can be investigated for more general matrices $A(s,z)$ arising from singular retarded delay differential equations. Also a generalization of the above results to more then two variables s and z can be considered.

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