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ON A THIRD ORDER PARABOLIC EQUATION WITH NONLOCAL BOUNDARY CONDITIONS

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Abstract

In this paper, we study a mixed problem for a third order parabolic equation with non classical boundary condition. We prove the existence and uniqueness of the solution. The proof of the uniqueness is based on a priori estimate and the existence is established by Fourier's method.

***Keywords*.**: Integral Boundary Condition, Energy Inequalities, Parabolic equation of mixed type.

Resumé

Dans cet article, nous étudions un problème mixte pour une équation parabolique du troisième ordre avec condition aux limites non classique. Nous démontrons l'existence et l'unicité de la solution. La preuve de la spécificité est basée sur une estimation a priori et de l'existence est établie par la méthode de Fourier.

Mots clés: Integral Boundary Condition, énergie Inégalités, l'équation parabolique de type mixte

ملخص

في هذه المقال، نحن ندرس مشكلة مختلطة للأمر الثالث معادلة القطع المكافئ مع حالة الحدود غير التقليدية. علينا أن نثبت وجود التفرد من الحل. ويستند هذا دليل على تفرد على تقدير مسبق وتثبت وجود بطريقة فورييه.

الكلمات المفتاحيـــة : لا يتجزأ الحدود الحالة، الطاقة عدم المساواة، معادلة مكافئ من النوع المختلط..

I - INTRODUCTION

In the set $\Omega = (0,T)\times(0,1)$, we consider the equation

$$\frac{\partial^2 u}{\partial t^2} - \frac{1}{x^2}\left(\frac{\partial}{\partial x}\left(x^3\frac{\partial^2 u}{\partial x\partial t}\right)\right) + k\frac{\partial u}{\partial t} = F(t,x), \quad \text{and} \quad k \geq 0, \tag{1.1}$$

To equation (1.1) we attach the initial conditions

$$u(0,x) = \varphi(x) \qquad x \in (0,1), \tag{1.2}$$

$$\frac{\partial u(0,x)}{\partial t} = \psi(x) \qquad x \in (0,1), \tag{1.3}$$

and the integral conditions

$$\int_0^1 u(t,x)dx = 0\,, \int_0^1 x^2 u(t,x)dx = 0 \quad \text{for} \quad t \in (0,T) \tag{1.4}$$

Where $\varphi(x), \psi(x) \in L_2(0,1)$ are known functions which satisfy the compatibility conditions given in (1.4).

The boundary value problems with integrals conditions are mainly motivated by the work of Samarskii [3]. Regular case of this problem for second order equations is studied in [4]. The problem where the equation of mixed type contains an operator of the form $a(t)\dfrac{\partial^{2\alpha+1}u}{\partial x^{2\alpha}\partial t}$ is treated in [17], the operator of the form $\dfrac{\partial}{\partial x}\left(a(t,x)\dfrac{\partial u}{\partial x}\right)$ and $\dfrac{\partial^\alpha}{\partial x^\alpha}\left(a(t,x)\dfrac{\partial^\alpha u}{\partial^\alpha x}\right)$ is treated in [4] and [14]. Two-point boundary value problems for parabolic equations, with an integral condition, are investigated using the energy inequalities method in [8, 9, 10, 11] and the Fourier's method [12]. Three-point boundary value problem with an integral condition for parabolic equations with the Bessel operator is studied in [12]. And recently parabolic and hyperbolic equations with integral boundary condition are treated by Fourier's method in [1, 5].

The presence of nonlocal conditions raises complications in applying standard methods to solve (1.1)-(1.4). Therefore to over come this difficulty we will transfer this problem to another which we can handle more effectly. For that, we have the following lemma.

Lemma 1. *Problem (1.1)-(1.4) is equivalent to the following problem*

$$(\mathrm{Pr})_1 \quad \begin{cases} \dfrac{\partial^2 u}{\partial t^2} - \dfrac{1}{x^2}\left(\dfrac{\partial}{\partial x}\left(x^3\dfrac{\partial^2 u}{\partial x\partial t}\right)\right) + k\dfrac{\partial u}{\partial t} = F(t,x) \\ u(0,x) = \varphi(x) \\ \dfrac{\partial u(0,x)}{\partial t} = \psi(x) \\ \dfrac{\partial u(t,1)}{\partial t} - \dfrac{\partial u(t,0)}{\partial t} = \dfrac{1}{2}\left(\int_0^1 x^2 F(t,x)dx - \int_0^1 F(t,x)dx\right) \\ \dfrac{\partial^2 u(t,1)}{\partial x\partial t} = -\int_0^1 x^2 F(t,x)dx \end{cases}$$

Proof. Let u(t,x) be a solution of (1.1)-(1.4). Integrating equation (1.1) with respect to x over (0,1), and taking into account of (1.4), we obtain

$$-\left[x\frac{\partial^2 u}{\partial x\partial t}\right]_0^1 - 2\int_0^1 \frac{\partial^2 u}{\partial x\partial t}dx = \int_0^1 F(t,x)dx$$

And so

$$\frac{\partial^2 u}{\partial x\partial t}(t,1) + 2\left(\frac{\partial u(t,1)}{\partial t} - \frac{\partial u(t,0)}{\partial t}\right) = -\int_0^1 F(t,x)dx$$

To eliminate the second nonlocal condition $\int_0^1 x^2 u(t,\xi)d\xi = 0$ multiplying both sides of (1.1) by x^2 and integrating the resulting over (0,1), and taking in account of (1.4), we obtain

$$\frac{\partial^2 u(t,1)}{\partial x\partial t} = -\int_0^1 x^2 F(t,x)dx$$

These may also be written

$$\frac{\partial u(t,1)}{\partial t} - \frac{\partial u(t,0)}{\partial t} = \frac{1}{2}\left(\int_0^1 x^2 F(t,x)dx - \int_0^1 F(t,x)dx\right)$$

And

$$\frac{\partial^2 u(t,1)}{\partial x\partial t} = -\int_0^1 x^2 F(t,x)dx$$

Let now $u(t,x)$ be a solution of $(\mathrm{Pr})_1$, it remains to prove that :

$$\int_0^1 u(t,x)dx = 0\,,$$

And

$$\int_0^1 x^2 u(t,x)dx = 0$$

We integrate Eq.(1.1) with respect to x, we obtain

$$\frac{d^2}{dt^2}\int_0^1 u(t,x)dx + k\frac{d}{dt}\int_0^1 u(t,x)dx = 0, \quad t \in (0,T)$$

And it also follows that

$$\frac{d^2}{dt^2}\int_0^1 x^2u(t,x)dx + k\frac{d}{dt}\int_0^1 x^2u(t,x)dx = 0, \quad t\in(0,T)$$

Introduce now the new function

$v(x,t) = u(x,t) - u_0(x,t)$, where

$$u_0(x,t) = \alpha(x)\int_0^t m_1(\tau)d\tau + \beta(x)\int_0^t m_2(\tau)d\tau, \alpha(x) = -x + x^2, \beta(x) = 2x - x^2,$$

$$m_1(t) = -\int_0^1 x^2F(t,x)dx, m_2(t) = \frac{1}{2}(-m_1(t) - \int_0^1 F(t,x)dx)$$

Then $(Pr)_1$ is transformed into the following problem

$$(Pr)_2\quad\begin{cases} \ell v \equiv \dfrac{\partial^2 v}{\partial t^2} - \dfrac{1}{x^2}\left(\dfrac{\partial}{\partial x}\left(x^3\dfrac{\partial^2 v}{\partial x\partial t}\right)\right) + k\dfrac{\partial v}{\partial t} = f(t,x) \\ lv = v(0,x) = \varphi(x) \\ qv = \dfrac{\partial v(0,x)}{\partial t} = \Psi(x) \\ \dfrac{\partial v(t,1)}{\partial t} = \dfrac{\partial v(t,0)}{\partial t} \\ \dfrac{\partial^2 v(t,1)}{\partial x\partial t} = 0 \end{cases}$$

Where

$$f(t,x) = F(t,x) + (\alpha(x) - \frac{\beta(x)}{2})\int_0^1 x^2F_t(t,x)dx + \frac{\beta(x)}{2}\int_0^1 F_t(t,x)dx + \gamma(t,x),$$

$$\gamma(t,x) = (-3+8x)m_1(t) + (6-8x)m_2(t) - k\alpha(x)m_1(t) - k\beta(x)m_2(t)$$

$$\Psi(x) = \psi(x) + \alpha(x)\int_0^1 x^2F(0,x)dx + \frac{\beta(x)}{2}(-\int_0^1 x^2F(0,x)dx + \int_0^1 F(0,x)dx)$$

2. A PRIORI ESTIMATE

we consider $(Pr)_2$ as a solution of the operator equation

$Lv = \mathscr{F}$, where $L = (\ell, l, q)$, $\mathscr{F} = (f, \varphi, \Psi)$.

The operator L is acting from the Banach space $D(L)=E$ to F where

$$E = \left\{ v : x^{\frac{3}{2}}v, x^{\frac{3}{2}}\frac{\partial v}{\partial t}, x^{\frac{3}{2}}\frac{\partial^2 v}{\partial x\partial t} \in L_2(0,1) \text{ and } x^{\frac{3}{2}}\frac{\partial v}{\partial t}, x^{\frac{3}{2}}\frac{\partial^2 v}{\partial x\partial t}, x^{\frac{3}{2}}\frac{\partial^2 v}{\partial t^2}, x^3\frac{\partial^3 v}{\partial x^2\partial t}, x^2\frac{\partial^2 v}{\partial x\partial t} \in L_2(\Omega^\tau) \right\}$$

With respect to the norm

$$\|v\|_E^2 = \int_{\Omega^\tau} x^3\left((\frac{\partial v}{\partial t})^2 + (\frac{\partial^2 v}{\partial x\partial t})^2 + (\frac{\partial^2 v}{\partial t^2})^2\right) + \int_{\Omega^\tau}\left(\frac{\partial}{\partial x}\left(x^3\frac{\partial^2 v}{\partial x\partial t}\right)\right)^2 + \sup_{0\le t\le T}\int_0^1 x^3\left(v^2 + (\frac{\partial v}{\partial t})^2 + (\frac{\partial^2 v}{\partial x\partial t})^2\right)$$

Here F is the Hilbert space with the norm

$$\|\mathscr{F}\|_F^2 = \|(f,\varphi,\psi)\|_F^2 = \int_{\Omega^\tau} f^2 + \int_0^1 (\Psi^2 + (\Psi')^2 + \varphi^2)$$

Lemma 1 *For any function $u \in E$, we have*

$$\frac{\exp(-cT)}{8}\int_0^1 x^2(v(\tau,x))^2 \le \frac{1}{8}\int_0^1 x^2\varphi^2 + \frac{1}{8}\int_0^1\int_0^\tau x^2(\frac{\partial v}{\partial t})^2$$

(2.1)

with the constant c satisfying $c \ge 1$.

Proof. Integrating by parts $(\exp(-ct)x^2v, \frac{\partial v}{\partial t})$ and using elementary inequalities yields (2.1).

Theorem 1. *For $(Pr)_2$ We have*

$$\|v\|_E \le C\|Lv\|_F,$$

Where $C \succ 0$ is independent on v

Proof. Let

$$Mv = 2x^2\frac{\partial v}{\partial t} + x^2\frac{\partial^2 v}{\partial t^2}$$

Consider the scalar product $(\ell v, Mv)$, and integrating over $\Omega^\tau = (0,\tau)\times(0,1)$, we get

$$(\ell v, Mv)_{L_2(\Omega^\tau)} = (1+\frac{k}{2})\int_0^1 x^2(\frac{\partial v}{\partial t}(\tau,x))^2 - (1+\frac{k}{2})\int_0^1 x^2\Psi^2 + 2k\int_{\Omega^\tau} x^2(\frac{\partial v}{\partial t})^2 + 2\int_{\Omega^\tau} x^3(\frac{\partial^2 v}{\partial x\partial t})^2 +$$
$$\int_{\Omega^\tau} x^2(\frac{\partial^2 v}{\partial t^2})^2 + \frac{1}{2}\int_0^1 x^3(\frac{\partial^2 v}{\partial x\partial t}(\tau,x))^2 - \frac{1}{2}\int_0^1 x^3(\Psi')^2 \geq (1+\frac{k}{2})\int_0^1 x^2(\frac{\partial v}{\partial t}(\tau,x))^2 - (1+\frac{k}{2})\int_0^1 x^2\Psi^2 +$$
$$2k\int_{\Omega^\tau} x^2(\frac{\partial v}{\partial t})^2 + 2\int_{\Omega^\tau} x^3(\frac{\partial^2 v}{\partial x\partial t})^2 + \int_{\Omega^\tau} x^2(\frac{\partial^2 v}{\partial t^2})^2 + \frac{1}{2}\int_0^1 x^3(\frac{\partial^2 v}{\partial x\partial t}(\tau,x))^2 - \frac{1}{2}\int_0^1 x^2(\Psi')^2$$

(2.2)

We now apply an ε -inequality to the term

$(\ell v, 2x^2\frac{\partial v}{\partial t} + x^2\frac{\partial^2 v}{\partial t^2})$ we obtain

$$(\ell v, 2x^2\frac{\partial v}{\partial t} + x^2\frac{\partial^2 v}{\partial t^2}) \leq \frac{1}{\varepsilon_1}\int_{\Omega^\tau} x^2 f^2 + \varepsilon_1\int_{\Omega^\tau} x^2(\frac{\partial v}{\partial t})^2 + \frac{1}{2\varepsilon_2}\int_{\Omega^\tau} x^2 f^2 + \frac{\varepsilon_2}{2}\int_{\Omega^\tau} x^2(\frac{\partial^2 v}{\partial t^2})^2 \quad (2.3)$$

From equation s we have

$$\frac{1}{8}\int_{\Omega^\tau}\left(\frac{\partial}{\partial x}\left(x^3\frac{\partial^2 v}{\partial x\partial t}\right)\right)^2 \leq \frac{1}{4}\int_{\Omega^\tau} x^2 f^2 + \frac{1}{4}\int_{\Omega^\tau} x^2(\frac{\partial^2 v}{\partial t^2})^2 + \frac{1}{4}k\int_{\Omega^\tau} x^2(\frac{\partial v}{\partial t})^2$$

(2.4)

Combining inequalities (2.1), (2.2), (2.3) and (2.4) and since $(x \leq 1)$ we obtain

$$\left(\frac{4\varepsilon_2 + 2\varepsilon_1 + \varepsilon_1\varepsilon_2}{4\varepsilon_1\varepsilon_2}\right)\int_{\Omega^\tau} f^2 + (1+\frac{k}{2})\int_0^1 \Psi^2 + \frac{1}{2}\int_0^1 (\Psi')^2 + \frac{1}{8}\int_0^1 \varphi^2$$
$$\geq$$
$$(1+\frac{k}{2})\int_0^1 x^3(\frac{\partial v}{\partial t}(\tau,x))^2 + (\frac{7k}{4} - \varepsilon_1 - \frac{1}{8})\int_{\Omega^\tau} x^2(\frac{\partial v}{\partial t})^2 + 2\int_{\Omega^\tau} x^3(\frac{\partial^2 v}{\partial x\partial t})^2 + (\frac{3}{4} - \frac{\varepsilon_2}{2})\int_{\Omega^\tau} x^2(\frac{\partial^2 v}{\partial t^2})^2 +$$
$$\frac{1}{2}\int_0^1 x^3(\frac{\partial^2 v}{\partial x\partial t}(\tau,x))^2 + \frac{1}{8}\int_{\Omega^\tau}\left(\frac{\partial}{\partial x}\left(x^3\frac{\partial^2 v}{\partial x\partial t}\right)\right)^2 + \frac{\exp(-cT)}{8}\int_0^1 x^3(v(\tau,x))^2$$

(2.5)

Next choosing $\varepsilon_i, i = 1,2$ as $\frac{7k}{4} - \varepsilon_1 - \frac{1}{8} = k_1 \succ 0$ and $\frac{3}{4} - \frac{\varepsilon_2}{2} = k_2 \succ 0$.

The left-hand side of (2.5) is independent of τ, hence replacing the right-hand side by its upper bound with respect to τ, in the interval $[0,T]$, we obtain the desired inequality.

This completes the proof.

3. EXISTENCE AND UNIQUENESS OF SOLUTION

WE shall establish the existence of solution of (Pr)$_2$.

For this we make use of the Fourier's method.

Consider the function $v_n(t,x) = T_n(t)X_n(t)$ where $X_n(t)$ is an eigenfunction of the BVP

$$\begin{cases} \frac{1}{x^2}\left(\frac{d}{dx}\left(x^3\frac{dX_n}{dx}\right)\right) - kX_n = \lambda_n X_n \\ X_n(1) = X_n(0) \\ \frac{dX_n}{dx}(1) = 0 \end{cases}$$

$\lambda_n, n = 1,2,......$ is called the eigenvalue corresponding to the eigenfunction $X_n(x)$, and $T_n(t)$ is satisfing the initial problem

$$\begin{cases} \frac{d^2T_n}{dt^2} - \lambda_n\frac{dT_n}{dt} = f_n(t) \\ T_n(0) = \varphi_n \\ \frac{dT_n}{dt}(0) = \Psi_n \end{cases}$$

$$\varphi(x) = \sum_{n=1}^{\infty}\varphi_n X_n(x)$$

$$\Psi(x) = \sum_{n=1}^{\infty}\Psi_n X_n(x)$$

$$\Psi'(x) = \sum_{n=1}^{\infty}\Psi^*{}_n X_n(x)$$

$$f(t,x) = \sum_{n=1}^{\infty} f_n(t) X_n(x)$$

And by the Parseval-Steklov equality

$$\|\varphi\|^2_{L_2(0,1)} = \sum_{n=1}^{\infty} \varphi_n^2,$$

$$\|\Psi\|^2_{L_2(0,1)} = \sum_{n=1}^{\infty} \Psi_n^2,$$

$$\|\Psi'\|^2_{L_2(0,1)} = \sum_{n=1}^{\infty} (\Psi_n^*)^2,$$

And $\int_0^1 f(t,x)dx = \sum_{n=1}^{\infty} f_n^2(t).$

Hence $\int_\Omega f^2(t,x)dxdt = \sum_{n=1}^{\infty} \int_0^T f_n^2(t).$

Then direct computation yields

$$T_n(t) = \varphi_n + \int_0^t \Psi_n \exp(\lambda_n \tau)d\tau + \int_0^t \int_0^s f_n(\tau)\exp(\lambda_n s - \lambda_n \tau)d\tau ds$$

$$\int_0^1 x^2 X_n(x) X_m(x)dx = 0, n \neq m$$

And

$$\varphi_n = \frac{\int_0^1 x^2 \varphi(x) X_n(x)dx}{\int_0^1 x^2 X_n^2(x)}$$

$$\Psi_n = \frac{\int_0^1 x^2 \Psi(x) X_n(x)dx}{\int_0^1 x^2 X_n^2(x)}$$

By principle of superposition, the solution of $(Pr)_2$ is given by the series

$$v(t,x) = \sum_{n=1}^{\infty} T_n(x) X_n(x). \qquad (3.1)$$

Then we have

Theorem 2. *Let* $f, \varphi \in L_2(\Omega)$, *and* $\Psi \in H^1(0,1)$. *Then the solution* $v(t,x)$ *of* $(Pr)_2$ *exists and is represented by series (3.1) which converges in E.*

Proof. Consider the partial sum $S_N(t,x) = \sum_{n=1}^{N} T_n(x) X_n(x)$ *of the series (3.1) then by theorem 1*

$$\left\| \sum_{n=1}^{N} T_n(x) X_n(x) \right\|_E^2 \leq C_1 \sum_{n=1}^{N} \left(\int_0^T f_n^2(t)dt + \varphi_n^2 + \Psi_n^2 + (\Psi_n')^2 \right) \qquad (3.2)$$

The series $\sum_{n=1}^{N} \int_0^T f_n^2(t)dt, \sum_{n=1}^{N} \varphi_n^2, \sum_{n=1}^{N} \Psi_n^2$, and $\sum_{n=1}^{N} (\Psi_n')^2$ converge. Therefore it follows from (3.2) that the series (3.1) converges in *E* and accordingly its sum $v \in E$.

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