DYNAMIC OF ONE DIMENSIONAL WAVE PACKET IN HIGH-ORDER APPROXIMATIONS OF NONLINEAR DISPERSION THEORY

Abstract

We are interested by the soliton state solutions of the higher order nonlinear Schrödinger equation which models the propagation of solitons in optical fibers. This nonlinear wave equation is solved by using the coupled amplitude-phase formulation. These gives rise to a coupled pair of equations, which describe the interaction and dynamics between the amplitude and the phase of the pulse. Integrating one of them, a characteristic equation is derived. For different particular cases of the dependent nonlinear parameters, various types of soliton solutions are investigated. In the absence of the third-order dispersion, we have obtained two different families of solitons: bright soliton in anomalous-dispersion regime and dark soliton in normal dispersion regime. Other family of bright solitons which is characterized by a simple quadratic dependence of the soliton phase on its amplitude, is obtained when the third-order dispersion effect is zero. It is specifically investigated the dynamics of solitons in the presence of third-order dispersion which is well described by the Korteweg-de Vries nonlinear equation.

Keywords: *High-order nonlinear Schrödinger equation, soliton, optical fiber.*
PACS numbers: 42.65.-k; 42. 65.Ky; 42.65.Tg.

Résumé

On s'intéresse aux solutions solitoniques de l'équation de Schrödinger non linéaire d'ordre supérieur, qui décrit la propagation des solitons dans les fibres optiques. Cette équation d'ondes non linéaire est résolue en employant la formulation de couplage amplitude-phase. Cela conduit à une paire d'équations couplées, qui décrivent l'interaction et la dynamique entre l'amplitude et la phase de l'impulsion. En intégrons une, on a dérivé une équation caractéristique. Pour différents cas particuliers des paramètres non linéaires dépendants, de divers types de solitons sont étudiés. En l'absence de la dispersion de troisième ordre, nous avons obtenu deux familles différentes des solitons: un soliton brillant dans le régime de dispersion anormal et un soliton sombre dans le régime de dispersion normal. Une autre famille des solitons brillant qui est caractérisée par une dépendance quadratique simple de la phase de soliton à l'égard son amplitude, est obtenue quand l'effet de troisième ordre de dispersion est nul. C'est spécifiquement étudié la dynamique des solitons en présence de la dispersion de troisième ordre qui est bien décrite par l'équation de Korteweg-de Vries non linéaire.

Mots clés: *Equation de Schrödinger non linéaire d'ordre supérieur, soliton, fibre optique.*

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It is well-known that the study of nonlinear equations and their solutions is of great importance in many areas of physics. Of particular importance are nonlinear wave equations, which admit large amplitude solitary wave, or soliton solutions. A soliton phenomenon is essentially a nonlinear phenomenon which exists due to a delicate equilibrium between the dispersion and the nonlinearity effects. During their propagation, such waves travel with constant configurations (shape, speed, *etc*.) as long as they do not meet any external perturbations. Moreover, when they meet another of its kinds, the interaction process happens in such a way there is no destroying in each identities. The study of solitary waves represents a very interesting and useful physical phenomenon, with applications in many fundamental areas of physics. Indeed, a great number of soliton excitations are reported in literature. For example, in theories of Bloch walls which separate domains in magnetic materials [1], structural phase transitions [2], liquid He^3 [3], Josephson transmission lines [4], and most recently in theory of the low temperature conductivity of one dimensional Fröhich charge-density-wave condensates [5].

There are three nonlinear equations which are known to have soliton solutions. The first one, which is also historically the first soliton equation, is the Korteweg-de Vries (KdV) equation; which was developed to study water waves in shallow canal [6]. The KdV equation may be used to approximate any system weakly nonlinear and weakly dispersive and for which the phase velocity is a quadratic function of the wavevector [7-9]. The second is the nonlinear Schrödinger equation (NLS). This equation arises in a variety of contexts in nonlinear optics, and can be

used, in general, to describe the propagation of weakly nonlinear, nearly monochromatic wave (*i.e.*, wave packets) in a dispersive medium [10, 11]. The important feature of this equation is that the relevant solutions are wave packets, *i.e.*, the envelope of rapidly varying signal is the solitary wave. The third equation is the Sine-Gordon (SG) [12, 13]. It possesses the standard sinusoidal solutions for small amplitude excitations (*i.e.*, phonons). It has, also, soliton solutions and bound soliton-antisoliton pairs termed "breathers". In other words, any solution of the SG equation may be expressed as a combination of solitons, breathers, and phonons.

For wave packets several wavelengths long $\left(\Delta \approx (2-3)\lambda\right)$, the study of high-order solitons becomes a necessity in the description of waves propagating in nonlinear and dispersive systems.

In this paper, high-order terms are introduced in the nonlinear equation for packet's envelope. By considering the amplitude and the phase of the complex solution separately, the nonlinear equation is converted into two coupled amplitude-phase equations. These equations describe the interaction and the dynamics between the amplitude and the phase of the propagating wave. Integrating one of them, a characteristic equation, linking the phase and the amplitude, is obtained. Many soliton solutions are determined by a suitable choice of the nonlinear parameters q, β, μ, and γ.

1- COUPLED AMPLITUDE-PHASE EQUATIONS

As it is well known the propagation of light pulses in optical fibres is described by the higher order nonlinear Schrödinger equation [14]:

$$2i\left(\frac{\partial \psi}{\partial t} + \beta|\psi|^2\frac{\partial \psi}{\partial \xi} + \mu\psi\frac{\partial|\psi|^2}{\partial \xi}\right) + q\frac{\partial^2 \psi}{\partial \xi^2} + i\gamma\frac{\partial^3 \psi}{\partial \xi^3} + \alpha|\psi|^2\psi = 0 \tag{1}$$

where q, β, μ, and γ are nonlinear parameters, $\psi(\xi, t)$ is the envelope of the optical signal depending on the propagation distance ξ and the time variable t. Note that the last three terms of Eq.(1) include, respectively, the effects of group velocity dispersion, third order dispersion and self-phase modulation.

Without loss of generality, the solution $\psi(\xi, t)$ to Eq.(1) is assumed to take on the form:

$$\psi(\xi, t) = A(\xi - Vt)\exp\left[i\Omega t + i\varphi(\xi - Vt)\right] \tag{2}$$

Here $A(\xi - Vt)$ and $\varphi(\xi - Vt)$ represent, respectively, the amplitude and the phase of the optical wave, depending on (ξ, t) and the soliton velocity V.

The coupled amplitude-phase equations are obtained using (2) into (1):

$$q\frac{d^2 A}{d\varepsilon^2} - qA\left(\frac{d\varphi}{d\varepsilon}\right)^2 + \alpha A^3 - 2\Omega A + 2A\left(V - \beta A^2\right)\left(\frac{d\varphi}{d\varepsilon}\right)$$
$$-3\gamma\left(\frac{d\varphi}{d\varepsilon}\right)\left(\frac{d^2 A}{d\varepsilon^2}\right) + \gamma A\left(\frac{d\varphi}{d\varepsilon}\right)^3 - \gamma A\frac{d^3\varphi}{d\varepsilon^3} - 3\gamma\left(\frac{dA}{d\varepsilon}\right)\left(\frac{d^2\varphi}{d\varepsilon^2}\right) = 0 \tag{3}$$

$$qA\frac{d^2\varphi}{d\varepsilon^2} + 2q\left(\frac{dA}{d\varepsilon}\right)\left(\frac{d\varphi}{d\varepsilon}\right) + 2\left[(\beta + 2\mu)A^2 - V\right]\frac{dA}{d\varepsilon}$$
$$+ \gamma\frac{d^3 A}{d\varepsilon^3} - 3\gamma\left(\frac{dA}{d\varepsilon}\right)\left(\frac{d\varphi}{d\varepsilon}\right)^2 - 3\gamma A\left(\frac{d\varphi}{d\varepsilon}\right)\left(\frac{d^2\varphi}{d\varepsilon^2}\right) = 0 \tag{4}$$

where $\varepsilon = \xi - Vt$.

It is clear that solving the coupled nonlinear equations (3 and 4) is equivalent to solve the nonlinear equation (1).

It is necessary to note that in a previous work [14], Eq.(1) has been studied in the case of steady-state waves with linear phase modulation $d\varphi/d\varepsilon = k = const$. Our strategy, in this approach, is to treat this equation in another fashion. First, we try to find the general amplitude-phase relation from the system of the two-coupled equations (3, 4). Second, for all the possible cases of the nonlinear parameters existing in these equations, the corresponding soliton solutions are investigated.

2- GENERAL PHASE-AMPLITUDE RELATION

Multiplying by A, Eq.(4) may be written in the form:

$$\left[qA^2\left(\frac{d\varphi}{d\varepsilon} - \frac{V}{q}\right)\right]_\varepsilon + \frac{\beta + 2\mu}{2}\left[A^4\right]_\varepsilon + \gamma\left[A\frac{d^2 A}{d\varepsilon^2}\right]_\varepsilon$$
$$-\frac{\gamma}{2}\left[\left(\frac{dA}{d\varepsilon}\right)^2\right]_\varepsilon - \frac{3\gamma}{2}\left[A^2\left(\frac{d\varphi}{d\varepsilon}\right)^2\right]_\varepsilon = 0$$

where the indices represent the derivative with respect to the coordinate ε.

The last equation can be immediately integrated to yield:

$$qA^2\left(\frac{d\varphi}{d\varepsilon} - \frac{V}{q}\right) + \frac{\beta + 2\mu}{2}A^4$$
$$+ \gamma\left(A\frac{d^2 A}{d\varepsilon^2} - \frac{1}{2}\left(\frac{dA}{d\varepsilon}\right)^2 - \frac{3}{2}A^2\left(\frac{d\varphi}{d\varepsilon}\right)^2\right) = 0 \tag{5}$$

where the constant of integration has been set to zero for the solitary wave solution.

Equation (5) represents, then, the general relation which links the phase and the amplitude for the propagating wave, in the case of high-order nonlinear and dispersion approximations theory. For different particular cases of nonlinear parameters q, β, μ, and γ, various soliton solutions can be obtained. They propagate in different systems which are both nonlinear and dispersive. Note these solutions have the remarkable property that they retain their shape during propagation and, hence, may be classified as solitary waves.

3- SOLITON STATES SOLUTIONS

According to the Eq.(5), we can discern various cases for the packet's phase φ as follows:

3.1- Case 1: $\beta = -2\mu$, $\gamma = 0$, $q \neq 0$

Under the above conditions, Eq.(5) yields

$$\frac{d\varphi}{d\varepsilon} = V' \tag{6}$$

where $V' = V/q$.

Therefore, the phase function $\varphi(\varepsilon)$ can be easily obtained by integrating Eq.(6):

$$\varphi(\varepsilon) = V'\varepsilon + \varphi_o \tag{7}$$

where φ_o is the initial phase at $\varepsilon = 0$.

Eq.(7) shows that the steady-state waves are characterized by a linear phase modulation.

According to the above conditions, Eq.(3) can be reduced to the following form:

$$q\frac{d^2 A}{d\varepsilon^2} - qA\left(\frac{d\varphi}{d\varepsilon}\right)^2 + \alpha A^3 + 2A\left(V - \beta A^2\right)\left(\frac{d\varphi}{d\varepsilon}\right) - 2\Omega A = 0 \tag{8}$$

By inserting (6) into (8), one obtains, for the amplitude function $A(\varepsilon)$, the *"characterized equation"* for the fundamental solitary waves:

$$\frac{d^2 A}{d\varepsilon^2} = \frac{2\Omega - V'^2 q}{q} A - \frac{\alpha - 2\beta V'}{q} A^3 \tag{9}$$

Multiplying by $2\left(dA/d\varepsilon\right)$, Eq.(9) yields

$$\left(A_\varepsilon^2\right)_\varepsilon = \frac{2\Omega - V'^2 q}{q}\left(A^2\right)_\varepsilon - \frac{\alpha - 2\beta V'}{2q}\left(A^4\right)_\varepsilon$$

which can be integrated to yield

$$\varepsilon - \varepsilon_o = \int \frac{dA}{\sqrt{\dfrac{2\beta V' - \alpha}{2q} A^4 - \dfrac{qV'^2 - 2\Omega}{q} A^2 + K}} \tag{10}$$

where K and ε_o are integration constants.

Eq.(10) is an elliptic integral with a constant K free to be chosen. Explicit analytical solutions are obtainable only for a very few specific cases. In the following sections, the value of K will be chosen so that the solution is physically meaningful.

3.1.1- Bright soliton in anomalous dispersion regime

Practically, it is of interest to consider the propagation of light pulses in dielectric fibres when the dispersion is anomalous. In this regime the dispersive parameter of the propagating waves is negative (*i.e.*, $\beta_2 = -\frac{1}{v_g}\frac{dv_g}{d\omega} < 0$).

This parameter is represented by q in Eq.(1). So we assume that the value of q is negative (*i.e.*, $q < 0$) which corresponds to the choice $K = 0$ in Eq.(10). In such a case, the integration of Eq.(10) results in an hyperbolic secant function for the amplitude $A(\varepsilon)$, which is the standard form for the fundamental "*bright soliton*":

$$\psi(\varepsilon,t) = \sqrt{\frac{2\left(qV'^2 - 2\Omega\right)}{2\beta V' - \alpha}}\, \sec h\left[\sqrt{-\frac{qV'^2 - 2\Omega}{q}}\left(\varepsilon - \varepsilon_o\right)\right] e^{i\,(\Omega t + V'\varepsilon + \varphi_o)} \tag{11}$$

Eq.(11) can be rewritten in an other form

$$\psi(\varepsilon,t) = \sqrt{a}\, \sec h\left[\sqrt{\frac{a}{\sigma}}\left(\varepsilon - \varepsilon_o\right)\right] e^{i\,(\Omega\, t + V'\varepsilon + \varphi_o)} \tag{12}$$

where

$$a = \sqrt{2\left(qV'^2 - 2\Omega\right)\big/\left(2\beta V' - \alpha\right)} \text{ and } \sigma = -2q\big/\left(2\beta V' - \alpha\right).$$

Here a is the input soliton power, a/σ represents the soliton shape scale and σ is the nonnegative ratio of dispersion to the nonlinear coefficients $(\alpha,\ \beta)$.

The solution (12) is seen to be a rapidly varying sinusoid whose envelope is a solitary wave decaying exponentially to zero for large ε . Figure 1 represents the evolution of bright soliton normalized intensity in the case of $a = 1$ and $\sigma = 1$.
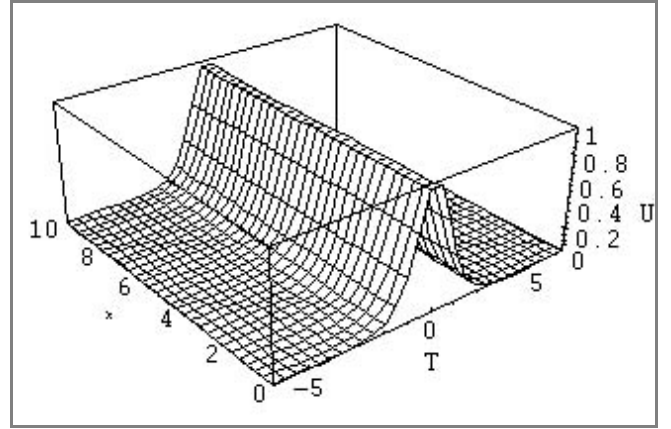

Figure 1: Evolution of bright soliton normalized intensity $u(X,T) = |\psi|^2/a$, as function of normalized distance $X = \xi\sqrt{a/\sigma}$ and normalized time $T = tV\sqrt{a/\sigma}$ in the case of a = 1 and σ = 1.

3.1.2- Dark soliton in normal dispersion regime

We consider now the conditions of a normal dispersion regime where the parameter q is positive (*i.e.*, $q > 0$). For this regime the dispersion parameter is positive (*i.e.*, $\beta_2 = -\frac{1}{v_g}\frac{dv_g}{d\omega} > 0$). This is guarantied when the integration constant K is chosen so that $K = \left[qV'^2 - 2\Omega\right]^2\big/2q(2\beta V' - \alpha)$ with the additional constraint $|A| < \sqrt{\left(qV'^2 - 2\Omega\right)\big/\left(2\beta V' - \alpha\right)}$ to make the quantity inside the square root sign in Eq.(10) a perfect square. Therefore, the integration results in a hyperbolic tangent function:

$$\psi(\varepsilon,t) = \sqrt{a}\, \tanh\left[\sqrt{-\frac{a}{\sigma}}\left(\varepsilon - \varepsilon_o\right)\right] e^{i\,(\Omega t + V'\varepsilon + \varphi_o)} \tag{13}$$

Therefore Eq.(13) describes the "*dark soliton*" solution, whereas Eq.(12) describes the "*bright soliton*" solution. Figure 2 depicts the evolution of dark soliton normalized intensity in the case of $a = 1$ and $\sigma = 1$.
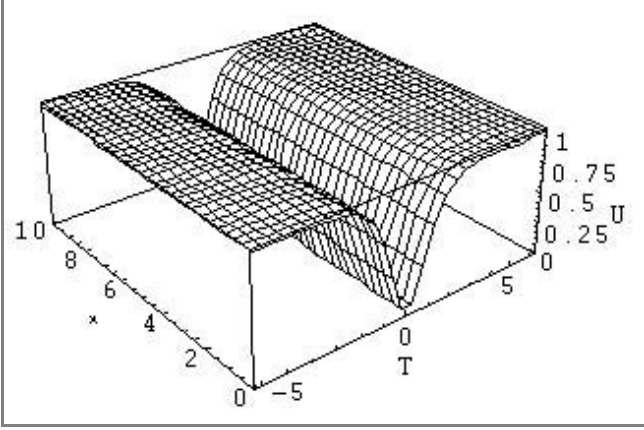
Figure 2: Evolution of dark soliton normalized intensity $u(X,T) = |\psi|^2/a$, as function of normalized distance $X = \xi\sqrt{a/\sigma}$ and normalized time $T = tV\sqrt{a/\sigma}$ in the case of $a = 1$ and $\sigma = 1$.

Two other analytical solutions are obtained when $|A| > \sqrt{\left((2-q)\,V'^2 - 2\Omega\right)/(2\beta V' - \alpha)}$. These are, respectively, the hyperbolic cotangent function and the secant function for $K = 0$:

$$\psi(\varepsilon,t) = -\sqrt{\frac{qV'^2 - 2\Omega}{2\beta V' - \alpha}}\coth\left[\sqrt{\frac{qV'^2 - 2\Omega}{2q}}(\varepsilon - \varepsilon_o)\right]e^{i\,(\Omega t + V'\varepsilon + \varphi_o)} \tag{14}$$

$$\psi(\varepsilon,t) = \sqrt{\frac{2\left(qV'^2 - 2\Omega\right)}{2\beta V' - \alpha}}\sec\left[\sqrt{\frac{qV'^2 - 2\Omega}{q}}(\varepsilon - \varepsilon_o)\right]e^{i\,(\Omega t + V'\varepsilon + \varphi_o)} \tag{15}$$

Those analytical solutions are non-physical due to the singularities in the solutions.

Remark: In a similar manner, Eq.(6) is obtained in the case of $\beta = \mu = \gamma = 0$. Under these conditions Eq.(1) takes the following form:

$$2i\frac{\partial\psi}{\partial t} + q\frac{\partial^2\psi}{\partial\xi^2} + \alpha|\psi|^2\,\psi = 0 \tag{16}$$

This last equation represents the well-known nonlinear Schrödinger equation which has been thoroughly studied in Refs.[15-17].

In conclusion, for this first particular case (3.1), we can say that our results, concerning the bright and dark solitons, are in good agreement with the references mentioned above. The only differences are in the input power a and shape scale a/σ for the soliton envelope expressions.

3.2- Case 2: $q \neq 0$, $\beta \neq$ -2μ, $\gamma = 0$

Under this conditions, the Eq.(5) yields

$$\frac{d\varphi}{d\varepsilon} = V' - \frac{\beta + 2\mu}{2q}A^2 \tag{17}$$

This relation shows a simple quadratic dependence of the phase $d\varphi/d\varepsilon$ with the amplitude A.

According to Eq.(17), the phase of the soliton is given by

$$\varphi(\varepsilon,t) = V'\varepsilon + \chi\int_0^\varepsilon A^2(\varepsilon')\,d\varepsilon' + \varphi_o \tag{18}$$

where $\chi = -(\beta + 2\mu)/2q$.

At the distance $\varepsilon = 0$, the time dependence of the phase is

$$\varphi(0,t) = \chi\int_0^\varepsilon A^2(\varepsilon')\,d\varepsilon' + \varphi_o \tag{19}$$

which is the initial phase distribution across the pulse.

From the above conditions, Eq.(3) is reduced to the form

$$q\frac{d^2 A}{d\varepsilon^2} - qA\left(\frac{d\varphi}{d\varepsilon}\right)^2 + \alpha A^3 - 2\Omega A + 2A\,\left(V - \beta A^2\right)\left(\frac{d\varphi}{d\varepsilon}\right) = 0 \tag{20}$$

Substituting Eq.(17) into Eq.(20) yields an equation for the packet's envelope A:

$$\frac{d^2 A}{d\varepsilon^2} = \frac{2\Omega - qV'^2}{q}A - \frac{\alpha - 2\beta V'}{q}A^3 - \frac{\beta + 2\mu}{q}\left(\beta - \frac{\beta + 2\mu}{4}\right)A^5 \tag{21}$$

which is a nonlinear ordinary differential equation.

By introducing the new variable z and the function B in such a way that:

$$z = \varepsilon\sqrt{\frac{2\Omega - V'^2 q}{q}}\;,\;\; B = A\sqrt{\frac{\alpha - 2\beta V'}{2\Omega - qV'^2}}$$

and using the new nonlinear parameter

$$\delta = (\beta + 2\mu)\left[\beta - \frac{\beta + 2\mu}{4}\right]\frac{2\Omega - qV'^2}{(\alpha - 2\beta V')^2}$$

Eq.(21) becomes

$$\frac{d^2 B}{dz^2} = B - B^3 - \delta\,B^5 \tag{22}$$

It is, also, a nonlinear ordinary differential equation with only one nonlinear parameter δ. We remark that both of Eq.(17) and Eq.(22) are similar to those deduced directly in Ref.[14] by assuming a linear phase modulation *i.e.,* $d\varphi/d\varepsilon = k = const$. Thus, the last equation corresponds to the variation of steady-state localized waves that are modelled by the modified nonlinear Schrödinger equation.

Multiplying Eq.(22) by $2(dB/d\varepsilon)$ yields

$$\left(B_\varepsilon^2\right)_\varepsilon = \left(B^2\right)_\varepsilon - \frac{1}{2}\left(B^4\right)_\varepsilon - \frac{1}{3}\delta\,\left(B^6\right)_\varepsilon$$

where the indices represent the derivative with respect to the coordinate ε.

By integrating this equation, one obtains:

$$\varepsilon - \varepsilon_o = \int\frac{dB}{\sqrt{B^2 - \frac{1}{2}B^4 - \frac{1}{3}\delta\,B^6}} \tag{23}$$

where the constant of integration is set to zero for solitary waves.

The soliton solution of Eq.(23) has the following form

$$B^2 = \frac{4}{1 + \sqrt{1 + 16\delta/3}\cosh(2z)} \tag{24}$$

As (24) implies, solitons exist if $(1 + 16\delta/3) > 0$.

The last equation shows that for large z, the envelope B tends to zero. This solution is a localized disturbance in which the phase depends quadraticaly on the amplitude of the wave (Eq.(17)). Figure 3 illustrates the dependence of soliton envelope B on the coordinate z for different values of the parameter δ: $\delta = 0$, $\delta = 9/16$, $\delta = 3/2$.

We can conclude, for the first and this second particular cases (3.1, 3.2), that our results are in good agreement with Refs.[13, 14].
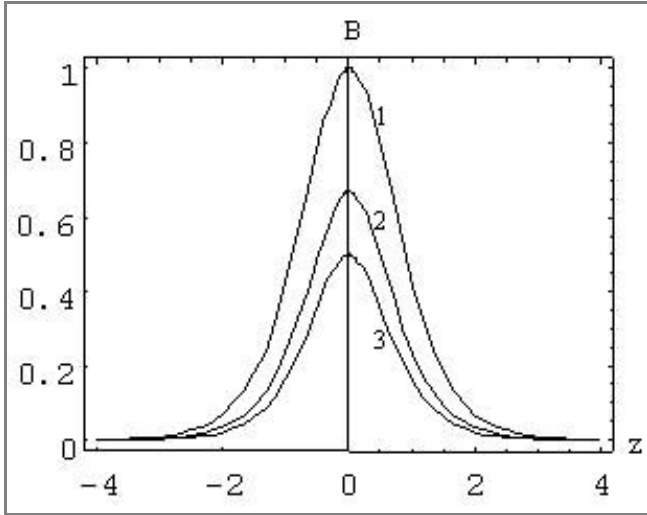

Figure 3: The dependence of the envelope B on the coordinate z for different values of the parameter δ.
Curves 1, 2 and 3 correspond, respectively, to the values δ=0, 9/16 and 3/2.

3.3- Case 3: $\theta = \beta = \mu = 0, \gamma \neq 0$

Using the above conditions, Eq.(5) yields two solution branches for the packet's phase φ given by

$$\frac{d\varphi}{d\varepsilon} = \pm \frac{1}{\sqrt{3}A} \sqrt{2A \frac{d^2 A}{d\varepsilon^2} - \left(\frac{dA}{d\varepsilon}\right)^2 - \frac{2V}{\gamma} A^2} \quad (25)$$

Thus, the phase of the soliton is given by

$$\varphi(\varepsilon,t) = \pm \frac{1}{\sqrt{3}} \int_0^\varepsilon \frac{1}{A} \sqrt{2A \frac{d^2 A}{d\varepsilon'^2} - \left(\frac{dA}{d\varepsilon'}\right)^2 - \frac{2V}{\gamma} A^2} \, d\varepsilon' + \varphi_o \quad (26)$$

where φ_o is the initial phase at $\varepsilon = 0$.

It is clear that obtaining the amplitude function A of the soliton, with the similar manner of the cases 4.1 and 4.2 (*i.e.,* by substituting (25) into (3) under the above conditions) is very difficult. But it can be easily determined by using directly Eq.(1) instead of Eq.(3).

Under the above conditions, Eq.(1) will be reduced in the case of $\alpha = 0$, to the form

$$2i \frac{\partial \psi}{\partial t} + i\gamma \frac{\partial^3 \psi}{\partial \xi^3} = 0 \quad (27)$$

Note that the linearized equation (27) describes the propagation of a wave disturbance in a purely dispersive system in which the dispersion is of the third order.

Multiplying by $\psi *$, with ψ is a real function $(\psi = \psi *)$, Eq.(27) becomes

$$\frac{\partial |\psi|^2}{\partial t} + \frac{\gamma}{2} \frac{\partial^3 |\psi|^2}{\partial \xi^3} - \frac{3\gamma}{2} \frac{\partial}{\partial \xi} \left(\frac{1}{4|\psi|^2} \left[\frac{\partial |\psi|^2}{\partial \xi} \right]^2 \right) = 0 \quad (28)$$

Using the variable $\varepsilon = \xi - Vt$, where V is the wave packet velocity, the integration of (28) gives

$$-Vu + \frac{\gamma}{2} \frac{d^2 u}{d\varepsilon^2} - \frac{3\gamma}{8} \frac{1}{u} \left(\frac{du}{d\varepsilon}\right)^2 = 0 \quad (29)$$

where $u = |\psi|^2$ is the soliton intensity.

Eq.(29) can be rewritten in the form

$$-V + \frac{\gamma}{2} \frac{u''}{u} = \frac{3\gamma}{8} \frac{u'^2}{u^2} \quad (30)$$

where u' and u'' are, respectively, the first and the second derivatives of the intensity u with respect to the coordinate ε.

By introducing the variable $v = u'/u$, Eq.(30) becomes

$$+V - \frac{\gamma}{8} v^2 = \frac{\gamma}{2} v' \quad (31)$$

which can be immediately integrated to yield

$$v(\varepsilon) = \sqrt{\frac{8V}{\gamma}} \tanh \left(\sqrt{\frac{V}{2\gamma}} (\varepsilon - \varepsilon_o) \right) \quad (32)$$

According to the transformation $v = u'/u$, the pulse intensity is therefore equal to

$$u(\xi,t) = \cosh^4 \sqrt{\frac{V}{2\gamma}} (\xi - Vt - \varepsilon_o) \quad (33)$$

where ε_o is an arbitrary constant of integration.

As a result, the amplitude function takes the form:

$$A(\xi,t) = \cosh^2 \sqrt{\frac{V}{2\gamma}} (\xi - Vt - \varepsilon_o) \quad (34)$$

Thus, the wave amplitude function (34), obtained in this case, have not the properties of a localized disturbance (*i.e.,* the function and its derivatives decay to zero at infinity). However, the presence of only dispersion term γ, in the starting equation (27), is insufficient to obtain a solitary wave solution.

3.4- Case 4: $\theta = \alpha = 0,\ \beta \neq -2\mu,\ \gamma \neq 0$

Under these conditions, Eq.(5) yields a complex relation for the packet's phase given by:

$$\frac{d\varphi}{d\varepsilon} = \pm \frac{1}{\sqrt{3}A} \sqrt{\frac{\beta + 2\mu}{\gamma} A^4 + 2A \frac{d^2 A}{d\varepsilon^2} - \left(\frac{dA}{d\varepsilon}\right)^2 - \frac{2V}{\gamma} A^2} \quad (35)$$

The packet's phase is therefore equal to

$$\varphi(\varepsilon,t) = \pm \frac{1}{\sqrt{3}} \int_0^\varepsilon \frac{1}{A} \sqrt{\frac{\beta + 2\mu}{\gamma} A^4 + 2A \frac{d^2 A}{d\varepsilon'^2} - \left(\frac{dA}{d\varepsilon'}\right)^2 - \frac{2V}{\gamma} A^2} \, d\varepsilon' + \varphi_o \quad (36)$$

In this case, and by multiplying Eq.(1) by $\psi *$ (where ψ is a real function), one obtains

$$\frac{\partial |\psi|^2}{\partial t} + (\beta + 2\mu)|\psi|^2 \frac{\partial |\psi|^2}{\partial \xi} + \frac{\gamma}{2} \frac{\partial^3 |\psi|^2}{\partial \xi^3} = \frac{3\gamma}{2} \frac{\partial}{\partial \xi}\left(\psi_\xi^{\ 2} \right) \quad (37)$$

Neglecting the term of dispersion in the right-hand side of the above equation, it will reduced to the form

$$u_t + \frac{\beta + 2\mu}{2}\left(u^2 \right)_\xi + \frac{\gamma}{2} u_{\xi\xi\xi} = 0 \qquad (38)$$

where $u = |\psi|^2$ is the wave intensity.

Eq.(38) represents the well-known Korteweg-de Vries equation.

Assuming a travelling wave solution of the form $u(\xi,t) = u(\xi - Vt) = u(\varepsilon)$, Eq.(38) can be integrated to give

$$-Vu + \frac{\eta}{2} u^2 + \frac{\gamma}{2} u_{\varepsilon\varepsilon} = C$$

where $\eta = \beta + 2\mu$ and C is a constant of integration. In particular, for the solitary wave solution, this constant is zero.

Multiplying by $\left(2\, u_\varepsilon \right)$, one obtains

$$-V\left(u^2 \right)_\varepsilon + \frac{\eta}{2}\left(u^3 \right)_\varepsilon + \frac{\gamma}{2}\left(u_\varepsilon^{\ 2} \right)_\varepsilon = 0$$

This can be integrated immediately to give

$$-Vu^2 + \frac{\eta}{2} u^3 + \frac{\gamma}{2} u_\varepsilon^{\ 2} = 0 \qquad (39)$$

where the constant of integration is set to zero for the same reason as before.

Eq.(39) may be rewritten as follows

$$\frac{\sqrt{\gamma/2}}{u\sqrt{V - \eta\, u/3}} du = \pm d\varepsilon$$

The last equation can be integrated to yield

$$u(\xi,\, t) = \frac{3V}{\eta} \sec h^2 \left\{ \sqrt{\frac{2V/\gamma}{2}} (\xi - Vt - \varepsilon_o) \right\}$$

This can be rewritten in the form

$$u(\varepsilon) = \frac{3V}{\eta} \sec h^2 \left\{ \sqrt{\frac{2V/\gamma}{2}} (\varepsilon - \varepsilon_o) \right\} \qquad (40)$$

where ε_o is an arbitrary constant of integration.

Eq.(40) is the solitary wave solution to the Korteweg-de Vries equation, and it possesses a number of properties that distinguish it from other types of waves. First, note that bit is indeed a "*solitary wave*", *i.e.,* a localized disturbance, since it decays to zero at infinity. Second, we note, also, that its width and velocity depend on the amplitude of the wave. This is a distinguishing characteristic of the Korteweg-de Vries solitary wave. Figure 4 illustrates the dependence of the soliton intensity $u(\varepsilon)$ on the coordinate ε.
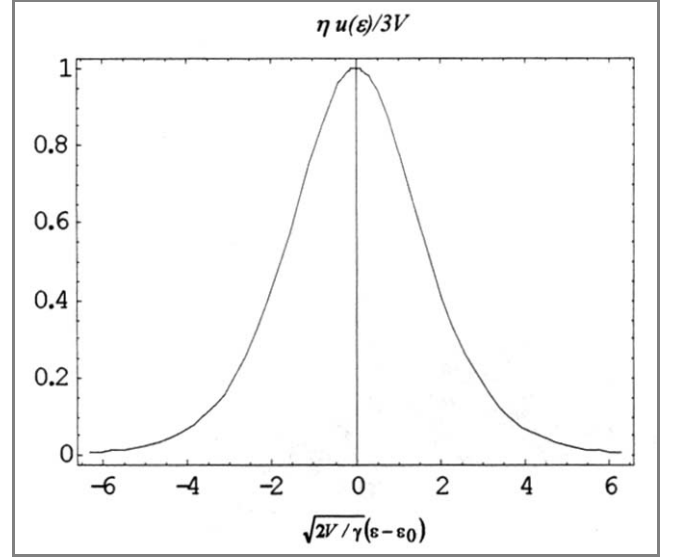

Figure 4: The dependence of the normalized intensity $\eta u\,(\varepsilon)/3V$ on the coordinate ε.

CONCLUSION

In this work, we have analyzed the nonlinear dynamics of optical solitary wave propagating in nonlinear and dispersive media. A soliton's general phase- amplitude relation has been obtained. Various cases of nonlinear parameter q, β, μ, and γ, are considered. For $\beta = -2\mu$, $\gamma = 0$ and $q \neq 0$, fundamental soliton solutions, for both anomalous dispersion regime ($q < 0$) and normal dispersion regime ($q > 0$) are derived. For other particular cases of the nonlinear parameters, new classes of steady-states nonlinear localized waves (solitons) are determined.

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