NUMERICAL METHOD FOR NON LOCAL PROBLEM

Reçu le 13/12/2007 – Accepté le 16/06/2008

A.CHENIGUEL*
A. AYADI**

*Departement of Mathematics and Computer Sciences
Faculty of Sciences
Kasdi Merbah University
Ouargla, Algeria
**Departement of mathematics and computer sciences
Faculty of sciences
Larbi Ben M'hidi University, Oum El Bouaghi, Algeria

Résumé

Dans ce travail on résout un problème parabolique avec des conditions aux limites non locales en utilisant la méthode des différences finis compactes d'ordre (d'ordre 6).la condition au limite intégrale est approchée la méthode de Simpson.les tests numériques montrent que la solution approchée coïncide avec la solution exacte sur plus de cinquante pour cent des points de discrétisation.

Mots clés: Schémas aux Différences Finis. Schémas Compactes d'ordre élevé, Problème non local, Ordre de convergence, Méthodes Numériques pour la résolution des équations aux dérivés partielles

Abstract

This paper is concerned with a high-order finite difference scheme for a non local boundary value problem of parabolic equation the integral in the boundary equation is approximated by the Simpson rule numerical experiments show that the approximate solution coincides with the exact one at more than fifty percent grid points discretization.

Keywords: Finite Difference Schemes, High-order Compact Schemes, Non local problem, Order of accuracy, Numerical methods for partial differential equations.

ملخص

نقدم في هذا العمل حلا تقريبيا لمعادلة تفاضلية ذات مشتقات جزئية بشروط حدّية غير محلّية مستعملين طريقة الفروق المنتهية المنضغطة من المرتبة السادسة . كما نقرّب الشرط الحدّي التكاملي مستعملين طريقة " سمسون " . إنّ الاختبارات العددية تبين أنّ الحلّ التقريبي المحصل عليه ينطبق على الحل الحقيقي في أكثر من خمسين بالمائة من نقاط التجزئة المقترحة .

الكلمات المفتاحيــــة :
صيغ الفروق المنتهية ، صيغ المنضغطة من مراتب عليا ، مسألة غير محلية ، مرتبة التقارب ، طرق عددية لحل المعادلات التفاضلية ذات المشتقات الجزئية

Introduction

In this Paper we first introduce the compact sixth-order finite difference formula then we adjust compact finite difference formula for the following heat equation with non local boundary conditions.

$$\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2}$$
$$(x,t)\in \left]0,1\right[\times\left]0,T\right[\quad (1)$$

$$u(x,0)=f(x) \quad 0<x<1 \quad (2)$$

$$u_x(1,t)=g(t) \quad 0<t<T \quad (3)$$

$$\int_0^b u(x,t)dx=m(t), \quad 0<t<T \quad (4)$$

where f , g , b and m are known.

J. Cannon and J. Vander Hoek [1] studied the existence and uniqueness proprieties of this problem. A.B. Gumel [2] has proposed numerical scheme of order $O(h_x^2+h_t^2)$ L_0-stable parallel algorithm for soling this Problem later M. Akram and pasha [3] have proposed a more accurate Algorithm of order $O(h_x^3+h_t^3)$. We propose a more accurate scheme of order $O(h_x^6+h_t^6)$.The numerical experiments show that the sixth-order schemes are unconditionally stable and more accurate than that in [3] furthermore for the choice $h_x=\frac{1}{8}$ and $h_t=\frac{1}{1000}$. The approximate solution coincides with the exact one at more than half of grid points discretization.

2 SIXT-ORDER COMPACT FINITE DIFFERENCE FORMULA

Compact formula is a special finite difference method which uses the values of the function and its derivatives only at three consecutive points.

First keeping time continuous, we carry out a spatial discretization of $\frac{\partial^2 u}{\partial x^2}$, we divide the interval [0,1] using a uniform grid $0=x_0<x_1<x_2<...<x_N$ with a mesh size $h_x=x_{i+1}-x_i$,

2.1 STANDARD COMPACT FINITE DIFFERENCE

The standard sixth-order compact finite difference formula for second derivative is:

$$\frac{h_x^2}{12}\left(U''_{i-1}(t)+10.U''_i(t)+U''_{i+1}(t)\right)= U_{i-1}(t)-2U_i(t)+U_{i+1}(t) \quad (5)$$

1. Write the compact finite difference formula in general form

$$h_x^2\left(a_{-1}U''_{i-1}(t)+a_0U''_i(t)+a_1U''_{i+1}(t)\right)= b_{-1}U_{i-1}(t)+b_0U_i(t)+b_1U_{i+1}(t) \quad (6)$$

where $a_{-1},a_0,a_1,b_{-1},b_0$ and b_1 are parameters to be determined.

2. Expand both sides of the equation (6) using Taylor series at the point x_i with respect to the discretization parameter h_x .

3. We obtain six equations by setting the coefficients h_x^j , $j=0,1,2,...,5$ equal zero solve the six equations for the six unknown parameters the obtained accuracy is $O(h^6)$ for formula (6).

2.2- WRITE EQUATION (1) IN A DISCRET POINT FORM

$$\frac{\partial u(x_i,t)}{\partial t}=\frac{\partial^2 u(x_i,t)}{\partial x_i^2} \quad i=1,,...,N-1. \quad (7)$$

Equation (5) is valid only for $i=2,,...,N-2$ to attain the same accuracy at $i=1$ and $i=N-1$ special formula must be developed.

When $i=1$ we use the formula

$$\frac{h_x^2}{12}\left(14U''_1(t)-5U''_2(t)+4U''_3(t)-U''_4(t)\right)= U_0(t)-2U_1(t)+U_2(t) \quad (8)$$

From Simpson integration rule we have

$$\int_0^b u(x,t)dx\approx\frac{h_x}{3}(u_0(t)+4u_1(t)+u_2(t))=m(t),$$

b has been chosen as a grid point, and when $i=N-1$ we use the formula

$$\frac{h_x^2}{12}\begin{pmatrix}\frac{-127}{30}U''_{N-4}(t)+\frac{86}{5}U''_{N-3}(t)- \\ U''_{N-2}(t)+\frac{461}{15}U''_{N-1}(t)\end{pmatrix} \quad (9)$$

$$=U_{N-2}(t)-U_{N-1}(t)+hU'_N(t)$$

We use U to stand for the approximation value of u throughout this paper.

All Formula are $O(h_x^6)$ or written in matrix form

$$AU''(t) = MU(t) + H \,, \tag{10}$$

where

$$A = \frac{h_x^2}{12}\begin{pmatrix} 14 & -5 & 4 & -1 & 0 & & & & & \\ 1 & 10 & 1 & 0 & & & & & & \\ 0 & 1 & 10 & 1 & 0 & & & & & \\ & 0 & 1 & 10 & 1 & 0 & & & & \\ & & 0 & 1 & 10 & 1 & 0 & & & \\ & & & 0 & 1 & 10 & 1 & 0 & & \\ & & & & 0 & 1 & 10 & 1 & 0 & \\ & & & & & 0 & 1 & 10 & 1 & 0 \\ & & . & & 0 & -\frac{127}{30} & \frac{86}{5} & -\frac{257}{10} & \frac{461}{15} & \end{pmatrix}$$

$$M = \begin{pmatrix} -6 & 0 & 0 & . & . & . & . & . & . \\ 1 & -2 & 1 & 0 & . & . & . & . & . \\ 0 & 1 & -2 & 1 & 0 & & & & \\ & 0 & 1 & -2 & 1 & 0 & & & \\ & & 0 & 1 & -2 & 1 & 0 & & \\ & & & 0 & 1 & -2 & 1 & 0 & \\ & & & & 0 & 1 & -2 & 1 & 0 \\ & & & & & 0 & 1 & -2 & 1 \\ & & & & & & 0 & 1 & -1 \end{pmatrix}$$

$$H = \begin{pmatrix} \frac{3}{h_x} m(t) \\ 0 \\ . \\ . \\ . \\ 0 \\ h_x U'_M \end{pmatrix}$$

Finally we obtain

$$U''(t) = A^{-1}MU(t) + A^{-1}H \,.$$

Putting $A^{-1}M = B$ and $A^{-1}H = R(t)$

$$U''(t) = BU(t) + R(t)\,. \tag{11}$$

Substituting in (7) we get a system of ordinary differential equations

$$\frac{dU(t)}{dt} = BU(t) + R(t)\,, \tag{12}$$

with the initial condition $U(0) = f(x)$.

Putting

$$f(t, U(t)) = BU(t) + R(t)\,,$$

we obtain the following equation

$$\frac{dU(t)}{dt} = f(t, U(t))\,. \tag{13}$$

We solve this equation using fourth-order Runge-Kutta Method

$$k_1 = f(t_0, U_0)$$

$$k_2 = f(t_0 + \frac{1}{2}h_t, \frac{k_1}{2}h_t U_0)$$

$$k_3 = f(t_0 + \frac{1}{2}h_t, \frac{k_2}{2}h_t U_0)$$

$$k_4 = f(t_0 + h_t, k_3 h_t + U_0)$$

$$U_{N+1} = U_N + \frac{1}{6}h_t(k_1 + 2k_2 + 2k_3 + k_4) \tag{14}$$

3 COMPUTATIONAL RESULTS

In order to test the sixth-order compact finite difference scheme, we consider the

Problem:

Consider the heat equation with

$$f(x) = 0.5x^2$$

$$g(t) = 1$$

$$m(t) = 0.75t + \frac{1}{6}(0.75)^3$$

which is easily seen to have exact solution

$$u(x,t) = 0.5x^2 + t\,.$$

Using Runge-Kutta method this problem is solved for

$$h_x = \frac{1}{8},$$

$$h_t = \frac{1}{1000} \quad ;$$

$$h_x = \frac{1}{10},$$

$$h_t = \frac{1}{1000}\,.$$

For $h_x = \frac{1}{8}$, $h_t = \frac{1}{1000}$, the results of approximate solution are tabulated in table 1.

x	*exact solution*	*approximate solution*	*absolute error*
1/8	8.8125×10^{-3}	8.3865×10^{-2}	7.5053×10^{-2}
1/4	0.03225	2.8182×10^{-2}	4.068×10^{-3}
3/8	7.1313×10^{-2}	7.1477×10^{-2}	1.64×10^{-4}
1/2	0.126	0.126	0
5/8	0.19631	0.19631	0
3/4	0.38381	0.38381	0
7/8	0.38381	0.38381	0

Table 1

For $h_x = \frac{1}{8}$, $h_t = \frac{1}{1000}$, the results of approximate solution are tabulated in table 2.

x	*exact solution*	*approximate solution*	*absolute error*
1/10	0.006	0.14449	0.13849
1/5	0.021	1.7083×10^{-2}	3.917×10^{-3}
3/10	0.046	4.5849×10^{-2}	1.51×10^{-4}
2/5	0.81	8.10446×10^{-2}	4.46×10^{-5}
1/2	0.126	0.126	0
3/5	0.181	0.181	0
7/10	0.246	0.246	0
4/5	0.321	0.321	0
9/10	0.406	0.406	0

Table 2

CONCLUSION

It is observed that the results obtained using compact sixth-order finite difference scheme are highly accurate. As compared to those of [3] and the method developed is sixth-order accurate in space and fourth-order in

Time with very high speed fourth-order Runge-Kutta algorithm it is to be noted that only one iterate was needed to obtain the results shown in both table 1 and 2.

REFERENCES

1. J. Cannon and J. Van Der Hoek, *The existence and a continuous dependence result for the heat equation subject to specification of energy*, Suppl Bolltino Unione Mat Hal, 1(1981), 253-282.

2. A. B. Gumel, *On the numerical solution of the diffusion equation subject to the speci- fication of Mass*, J. Auster. Math. Soc, Ser.B, 40(1999), 475-483.

3. M. Akram and M. A. Pasha, *A numerical method for the heat equation with a non local boundary condition*, International Journal of information and systems sciences, Volume1, Number 2(2005), 162-171.

4. J. D. Lambart, *Numerical Methods for ordinary differential systems*, The initial value problem, John Wiley and Sons, Chichester,1991.

5. Jichao Zhao* and Robert M. Corless**, *Compact finite difference Method for integro- differential equations*, (*) Department of applied mathematics, University of Western Ontario, Middlesex College, London, (**) Department of applied mathematics, University of Western Ontario, Middlesex College, London.

6. R. Cont, P. Tankov and E. Voltchkova, *Option pricing models with jumps, integro-differential equations and inverse problems*. European Congress on computational methods in applied sciences and Engineering, 24-28 July (2004), 1-24.

7. R. M. Corless and J. Rokieki, *The symbolic generation of finite difference formulas*, Zeitschrift fur Angwandte Mathmatik und Mechanik (Zamm), Proceeding, Iciam/Gamm 95, Numerical analysis, Scientific computing, Computer science, Hamburg, Germany, 3 July 1995, G. Alfred and O. mahrenholtz and R. Mennicken, ZAAM, Vol. 76 supplement, Germany Akademie Verlag, 381-382.

8. R.H.Li,Z. Y. Chen and W. Wu, *Generalized difference methods of differential equations, Analysis of finite volume methods*, Marcel Dekker Inc, New York, 2000.

9. H. Sun and J. Zhang, *A Highly Accurate derivative Recovery Formula to Integro-Differential Equations*, Numerical Mathematics, a journal of Chinese Universities, 2004, Vol 26(1), pp. 81-90.

10. M. O. Ahmed, *An exploration of compact finite difference methods for the numerical solution of PDE*, Ph.D. thesis, The University of Western Ontario, 1997.

11. M. S. A. Taj and E. H. Twizell, *A family of fourth order parallel splitting methods for parabolic partial differential equations*, Inc. Numer. Methods Partial differential eq. 13(1997), 357-373.