Sciences & Technologie A – N°46, Décembre 2017, 15-18

A THIRD ORDER HYPERBOLIC EQUATION WITH NONLOCAL CONDITIONS.

N. TEYAR

Department of Mathematics, University Frères Mentouri Constantine, 25000 Constantine, Algeria

Reçu le 03/01/2017 – Accepté le 12/11/2017

Abstract

In this paper, we study a mixed problem for a third order hyperbolic equation with non classical boundary condition. We prove the existence and uniqueness of the solution. The proof of the uniqueness is based on a priori estimate and the existence is established by Fourier's method.

Keywords: *nonlocal Boundary Condition, Energy Inequalities, hyperbolic equation of mixed type.*

Résumé

Dans cet article, on étudie un problème mixte pour une equation du troisième ordre avec des conditions non locales. On démontre l'unicité de la solution à partir d'une estimation a priori, et l'existence par la méthode standard de Fourrier.

Mots clés *: Conditions aux limites non locales, inégalités énergétiques, equation hyperbolique de type mixte*.

ملخص

في هذه المقالة، ندرس مسألة مختلطة لمعادلة تفاضلية زائدية المقطع من الرتبة الثالثة وبشروط غير محلية. نبرهن وحدانية الحل من خلال تقدير مسبق، ونثبت وجود الحل عبر منهج فوريي.

الكلمات المفتاحية : شروط الحدود غير المحلية ، عدم المساواة النشطة ، المعادلة الزائغة من النوع المختلط.

INTRODUCTION

In the set $\Omega=(0,T)\times(0,1)$, we consider the equation

$$\frac{\partial^3 u}{\partial t^3}-\frac{1}{x^2}\left(\frac{\partial}{\partial x}\left(x^4\frac{\partial^2 u}{\partial x\partial t}\right)\right)+k\frac{\partial^2 u}{\partial t^2}=F(t,x), \quad \text{and}$$

$$k\succ 0, \qquad (1.1)$$

To equation (1.1) we attach the initial conditions:

$$u(0,x)=\varphi(x) \qquad x\in(0,1), \qquad (1.2)$$

$$\frac{\partial u(0,x)}{\partial t}=\psi(x) \qquad x\in(0,1), \qquad (1.3)$$

$$\frac{\partial^2 u(0,x)}{\partial t^2}=\theta(x) \qquad x\in(0,1), \qquad (1.4)$$

and the integral conditions :

$$\int_0^1 u(t,x)dx=0\,,\ \int_0^1 x^2u(t,x)dx=0 \qquad \text{for}$$

$$t\in(0,T) \qquad (1.5)$$

Where $\varphi(x),\psi(x),\theta(x)\ \in L_2(0,1)$ are known functions which satisfy the compatibility conditions given in (1.5).

Integral boundary conditions for evolution problems have various applications in chemical engineering, thermoelasticity, underground water flow and population dynamics.

The boundary value problems with integrals conditions are mainly motivated by the work of Samarskii [7]. Two-point boundary value problems for parabolic equations, with an integral condition, are investigated using the energy inequalities method in [3, 4, 5, 8]. And recently parabolic and hyperbolic equations with integral boundary condition are treated by Fourier's method in [1, 2].

The presence of nonlocal conditions raises complications in applying standard methods to solve (1.1)-(1.5). Therefore to over come this difficulty we will transfer this problem to another which we can handle more effectly. For that, we have the following lemma.

Lemma 1. *Problem (1.1)-(1.5) is equivalent to the following problem*

$$(\mathrm{Pr})_1 \begin{cases} \dfrac{\partial^3 u}{\partial t^3}-\dfrac{1}{x^2}\left(\dfrac{\partial}{\partial x}\left(x^4\dfrac{\partial^2 u}{\partial x\partial t}\right)\right)+k\dfrac{\partial^2 u}{\partial t^2}=F(t,x) \\ u(0,x)=\varphi(x) \\ \dfrac{\partial u}{\partial t}(0,x)=\psi(x) \\ \dfrac{\partial^2 \mathrm{u}}{\partial t^2}(0,\mathrm{x})=\theta(\mathrm{x}) \\ \dfrac{\partial u}{\partial t}(t,1)=\dfrac{1}{2}\displaystyle\int_0^1 (x^2-1)F(t,x)dx \\ \dfrac{\partial^2 u}{\partial x\partial t}(t,1)=-\displaystyle\int_0^1 x^2F(t,x)dx \end{cases}$$

Proof. Let u(t,x) be a solution of (1.1)-(1.5). Integrating equation (1.1) with respect to x over (0,1), and taking into account of (1.5), we obtain

$$-\left[x^2\frac{\partial^2 u}{\partial x\partial t}\right]_0^1-2\int_0^1 x\frac{\partial^2 u}{\partial x\partial t}dx=\int_0^1 F(t,x)dx$$

And so

$$\frac{\partial^2 u}{\partial x\partial t}(t,1)+2\frac{\partial u(t,1)}{\partial t}=-\int_0^1 F(t,x)dx$$

To eliminate the second nonlocal condition $\int_0^1 x^2u(t,\xi)d\xi=0$ multiplying both sides of (1.1) by x^2 and integrating the resulting over (0,1), and taking in account of (1.5), we obtain:

$$\frac{\partial^2 u}{\partial x\partial t}(t,1)=-\int_0^1 x^2F(t,x)dx$$

These may also be written:

$$\frac{\partial u}{\partial t}(t,1)=\frac{1}{2}\int_0^1 (x^2-1)F(t,x)dx$$

And

$$\frac{\partial^2 u(t,1)}{\partial x\partial t}=-\int_0^1 x^2F(t,x)dx$$

Let now $u(t,x)$ be a solution of $(\mathrm{Pr})_1$, it remains to prove that:

$$\int_0^1 u(t,x)dx=0\,,$$

And

$$\int_0^1 x^2u(t,x)dx=0$$

We integrate Eq.(1.1) with respect to x, we obtain:

$$\frac{d^3}{dt^3}\int_0^1 u(t,x)dx+k\frac{d^2}{dt^2}\int_0^1 u(t,x)dx=0,\quad t\in(0,T)$$

And it also follows that:

$$\frac{d^3}{dt^3}\int_0^1 x^2u(t,x)dx+k\frac{d^2}{dt^2}\int_0^1 x^2u(t,x)dx=0,\ t\in(0,T)$$

So we have :

$$\int_0^1 u(t,x)dx=c_1+c_2t+c_3e^{-kt},\int_0^1 x^2u(t,x)dx=d_1+d_2t+d_3e^{-kt}$$

By virtue of the compatibility conditions, we get:

$$\int_0^1 u(t,x)dx=0 \text{ and } \int_0^1 x^2u(t,x)dx=0$$

Introduce now the new function:

$v(x,t)=u(x,t)-u_0(x,t),$ where

$$u_0(x,t)=\alpha(x)\int_0^t m_1(\tau)d\tau+\beta(x)\int_0^t m_2(\tau)d\tau,\alpha(x)=-x^2+x^3,\beta(x)=3x^2-2x^3,$$

$$m_1(t)=-\int_0^1 x^2F(t,x)dx,m_2(t)=\frac{1}{2}\int_0^1 (x^2-1)F(t,x)dx$$

Then $(\mathrm{Pr})_1$ is transformed into the following problem

$$(\mathrm{Pr})_2 \begin{cases} \ell v\equiv\dfrac{\partial^3 v}{\partial t^3}-\dfrac{1}{x^2}\left(\dfrac{\partial}{\partial x}\left(x^4\dfrac{\partial^2 v}{\partial x\partial t}\right)\right)+k\dfrac{\partial^2 v}{\partial t^2}=f(t,x) \\ lv=v(0,x)=\varphi(x) \\ pv=\dfrac{\partial v}{\partial t}(0,x)=\Psi(x) \\ qv=\dfrac{\partial^2 v}{\partial^2 t}(0,x)=\Theta(x) \\ \dfrac{\partial v}{\partial t}(t,1)=0 \\ \dfrac{\partial^2 v}{\partial x\partial t}(t,1)=0 \end{cases}$$

Where

$$f(t,x)=F(t,x)+\alpha(x)\int_0^1(kF_t(t,x)+F_{tt}(t,x))dx+\frac{\beta(x)}{2})\int_0^1(1-x^2)(kF_t(t,x)+F_{tt}(t,x))dx+\gamma(t,x),$$

$$\gamma(t,x)=(-10x^2+18x^3)m_1(t)+(30x^2-6x^3)m_2(t)$$

$$\Psi(x)=\psi(x)+\alpha(x)\int_0^1 x^2F(0,x)dx+\frac{\beta(x)}{2}\int_0^1(1-x^2)F(0,x)dx$$

$$\Theta(x)=\theta(x)+\alpha(x)\int_0^1 x^2F_t(0,x)dx+\frac{\beta(x)}{2}\int_0^1(1-x^2)F_t(0,x)dx$$

2. A PRIORI ESTIMATE:

we consider $(Pr)_2$ as a solution of the operator equation

$Lv = \mathscr{F}$,

where $L = (\ell, l, p, q)$, $\mathscr{F} = (f, \varphi, \Psi, \Theta)$. The operator L is acting from the Banach space $D(L)=E$ to F where space $D(L)=E$ to F where:

$$E = \left\{ v : x^2 \frac{\partial^2 v}{\partial t^2}, x^2 \frac{\partial^2 v}{\partial x \partial t} \in L_2(0,1) \text{ and } x \frac{\partial^2 v}{\partial t^2} \in L_2(\Omega^\tau) \right\}$$

With respect to the norm

$$\|v\|_E^2 = \int_{\Omega^\tau} x^2 \left|\frac{\partial^2 v}{\partial t^2}\right|^2 dxdt + \sup_{0\le\tau\le T} \int_0^1 x^4 \left(\left|\frac{\partial^2 v}{\partial t^2}(\tau,x)\right|^2 + \left|\frac{\partial^2 v}{\partial x \partial t}(\tau,x)\right|^2 \right) dx$$

Here F is the Hilbert space with the norm

$$\| \mathscr{F} \|_F^2 = \left\|(f, \varphi, \psi, \Theta)\right\|_F^2 = \int_{\Omega^\tau} f^2 + \int_0^1 \Psi^2 + (\Psi')^2$$

Theorem 1. *For $(Pr)_2$ We have*

$$\|v\|_E \le C \|Lv\|_F,$$

Where $C \succ 0$ is independent on v

Proof. Let $\qquad Mv = 2x^2 \dfrac{\partial^2 v}{\partial t^2}$

Consider the scalar product $(\ell v, Mv)$, and integrating over $\Omega^\tau = (0,\tau) \times (0,1)$, we get

$$(\ell v, Mv)_{L_2(\Omega^\tau)} == \int_0^1 x^2 \left|\frac{\partial^2 v}{\partial t^2}(\tau,x)\right|^2 dx - \int_0^1 x^2 |\Theta(x)|^2 dx + 2k \int_{\Omega^\tau} x^2 \left|\frac{\partial^2 v}{\partial t^2}\right|^2 dxdt + \int_0^1 x^4 \left|\frac{\partial^2 v}{\partial x \partial t}(\tau,x)\right|^2 dx - \int_0^1 x^4 |\Psi'(x)|^2 dx \tag{2.1}$$

We now apply an ε-inequality to the term $(\ell v, 2x^2 \dfrac{\partial^2 v}{\partial t^2})$ we obtain:

$$(\ell v, 2x^2 \frac{\partial^2 v}{\partial t^2}) \le \frac{1}{\varepsilon_1} \int_{\Omega^\tau} x^2 f^2 + \varepsilon_1 \int_{\Omega^\tau} x^2 (\frac{\partial^2 v}{\partial t^2})^2 \tag{2.2}$$

Combining equality (2.1) and inequality (2.2), and since $(x \le 1)$ we obtain:

$$\int_0^1 x^2 \left|\frac{\partial^2 v}{\partial t^2}(\tau,x)\right|^2 dx + (2k - \varepsilon_1) \int_{\Omega^\tau} x^2 \left|\frac{\partial^2 v}{\partial t^2}\right|^2 dxdt + \int_0^1 x^4 \left|\frac{\partial^2 v}{\partial x \partial t}(\tau,x)\right|^2 dx \le \frac{1}{\varepsilon_1} \int_{\Omega^\tau} f^2 + \int_0^1 |\Theta(x)|^2 dx - \int_0^1 |\Psi'(x)|^2 dx \tag{2.3}$$

Next choosing ε_1 as $2k \succ \varepsilon_1$.

The left-hand side of (2.3) is independent of τ, hence replacing the right-hand side by its upper bound with respect to τ, in the interval $[0,T]$, we obtain the desired inequality.

This completes the proof.

3. EXISTENCE AND UNIQUENESS OF SOLUTION

WE shall establish the existence of solution of $(Pr)_2$. For this we make use of the Fourier's method.

Consider the function $v_n(t,x) = T_n(t) X_n(t)$ where $X_n(t)$ is an eigenfunction of the BVP

$$\begin{cases} \dfrac{1}{x^2}\left(\dfrac{\partial}{\partial x}\left(x^4 \dfrac{\partial^2 v}{\partial x \partial t}\right)\right) - kX_n = \lambda_n X_n \\ X_n(1) = X_n(0) \\ \dfrac{\partial X_n}{\partial x}(1) = 0 \end{cases}$$

$\lambda_n, n = 1,2,\ldots\ldots$ is called the eigenvalue corresponding to the eigenfunction $X_n(x)$, and $T_n(t)$ is satisfying the initial problem

$$\begin{cases} \dfrac{d^3 T_n}{dt^3} - \lambda_n \dfrac{d^2 T_n}{dt^2} = f_n(t) \\ T_n(0) = \varphi_n \\ \dfrac{dT_n}{dt}(0) = \Psi_n \\ \dfrac{d^2 T_n}{dt^2}(0) = \Theta_n \end{cases}$$

$$\varphi(x) = \sum_{n=1}^{\infty} \varphi_n X_n(x)$$

$$\Psi'(x) = \sum_{n=1}^{\infty} \Psi^*{}_n X_n(x)$$

$$\Theta(x) = \sum_{n=1}^{\infty} \Theta_n X_n(x)$$

$$f(t,x)=\sum_{n=1}^{\infty} f_n(t)X_n(x)$$

And by the Parseval-Steklov equality

$$\|\varphi\|_{L_2(0,1)}^2=\sum_{n=1}^{\infty}\varphi_n^2,$$

$$\|\Psi\|_{L_2(0,1)}^2=\sum_{n=1}^{\infty}\Psi_n^2,$$

$$\|\Psi'\|_{L_2(0,1)}^2=\sum_{n=1}^{\infty}(\Psi_n^*)^2,$$

$$\|\Theta\|_{L_2(0,1)}^2=\sum_{n=1}^{\infty}\Theta_n^2,$$

And $$\int_0^1 f(t,x)dx=\sum_{n=1}^{\infty} f_n^2(t).$$

Hence $$\int_\Omega f^2(t,x)dxdt=\sum_{n=1}^{\infty}\int_0^T f_n^2(t).$$

Then direct computation yields:

$$T_n(t)=\varphi_n+\Psi_n t+\Theta_n\left(\frac{\exp(\lambda_n t)}{\lambda_n}-\frac{1}{\lambda^2{}_n}-t\right)+\int_0^t\int_0^\ell\int_0^s f_n(\tau)\exp(\lambda_n s-\lambda_n\tau)d\tau ds d\ell$$

$$\int_0^1 x^2 X_n(x)X_m(x)dx=0, n\neq m$$

And

$$\varphi_n=\frac{\int_0^1 x^2\varphi(x)X_n(x)dx}{\int_0^1 x^2X_n^2(x)dx}$$

$$\Psi_n=\frac{\int_0^1 x^2\Psi(x)X_n(x)dx}{\int_0^1 x^2X_n^2(x)dx}$$

$$\Theta_n=\frac{\int_0^1 x^2\Theta(x)X_n(x)dx}{\int_0^1 x^2X_n^2(x)dx}$$

By principle of superposition, the solution of $(Pr)_2$ is given by the series:

$$v(t,x)=\sum_{n=1}^{\infty} T_n(t)X_n(x). \qquad (3.1)$$

Then we have:

Theorem 2. *Let* $f,\Theta\in L_2(\Omega)$, *and* $\Psi\in H^1(0,1)$.

Then the solution $v(t,x)$ *of* $(Pr)_2$ exists and is represented by series *(3.1)* wich converges in *E*.

Proof. Consider the partial sum $S_N(t,x)=\sum_{n=1}^{N} T_n(t)X_n(x)$ of the series *(3.1)* then by theorem 1.

$$\left\|\sum_{n=1}^{N} T_n(t)X_n(x)\right\|_E^2\leq C_1\sum_{n=1}^{N}\left(\int_0^T f_n^2(t)dt+\Theta_n^2+(\Psi_n')^2\right) \qquad (3.2)$$

The series $\sum_{n=1}^{N}\int_0^T f_n^2(t)dt, \sum_{n=1}^{N}\Theta_n^2,$ and $\sum_{n=1}^{N}(\Psi_n')^2$ converge. Therefore, it follows from (3.2) that the series (3.1) converges in *E* and accordingly its sum $v\in E$.

REFERENCES

[1] M.A.Al-kadhi, A class of hyperbolic equations with nonlocal conditions, Int. J. Math. Analysis, 2 (10) (2008), 491-498.

[2] L.Bougoffa, Parabolic equations with nonlocal conditions, Appl. Math. Sciences 1 (21) (2007), 1041-1048.

[3] A. Bouziani and N. E. Benouar, Problème mixte avec conditions intégrales pour une classe d.équations paraboliques, C. R. Acad. Sci. Paris, Série 1 321 (1995), 1182.

[4] Bouziani, A. and Benouar, N.E., Mixed problem with integral conditions for a third order parabolic equation, Kobe J. Math. 15 (1998), 47-58.

[5] M. Bouzit, N. Teyar, On a Third Parabolic Equation with Nonlocal Boundary Conditions, sciences et technologies Université Frères Mentouri -Constantine N° 33 Juin 2011.

[6] M. Bouzit and N. Teyar ,A strong solution of a high-order mixed type differential equations with integral Boundary conditions, Advances in Differen- tial Equations and Control Processes, 2008, Volume 2, Issue 1, Pages 75 - 84.

[7] M. Bouzit, N. Teyar, A Class of Third Order Parabolic Equations with Integral Conditions, Int. Journal of Math. Analysis -2009.- Vol. 3, 18. - pp.871 - 877.

[8] A. A. Samarski, Some problems in differential equations theory, differ. Uravn, 16(11) (1980), 1221-1228

[9] N. I. Yurchuk, Mixed problem with an integral condition for certain parabolic equations, Differ. Equ. 22 (12), (1986),1457-1463.