SOLVING DIFFERENTIAL ALGEBRAIC EQUATIONS FOR SLIDER-CRANK MECHANISMS

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Abstract

The paper deals with the simulation of dynamical models of constrained multibody systems which can be formulated as a set of Differential Algebraic Equations (DAEs). We consider the general dynamical model resulting from Euler-Lagrange formulation. We investigate the solution with the index reduction method. We illustrate our analysis by the example of the slider-crank mechanism. Among many possible representations, we derive a dynamical model based on two variables offering an easier analysis and implementation. We solve the DAE problem with a Matlab function (ode15s) which is dedicated to solve stiff Ordinary Differential Equations (ODEs). Concordant simulation results have been obtained in comparison to other methods proving an acceptable stability and accuracy of the used method for solving this problem.

Mathematics Subject Classification (2000). Primary 70E55; Secondary 68U01.

Keywords : *DAE, ODE, Euler-Lagrange systems, dynamic modeling, multibody systems.*

Résumé

Cet article traite de la simulation de modèles dynamiques des systèmes multicorps contraints qui peuvent être formulés comme un ensemble d'équations différentielles algébriques (EDAs). Nous considérons le modèle dynamique général résultant de la formulation d'Euler-Lagrange. Nous étudions la solution avec la méthode de réduction de l'index. Nous illustrons notre analyse par l'exemple du mécanisme bielle-manivelle. Nous déterminons un modèle dynamique basé sur deux variables offrant une analyse et une implémentation aisée. Nous résolvons le problème DAE avec une fonction de Matlab (ode15s) qui est dédié pour résoudre équations différentielles ordinaires (EDO) raides.
Les résultats de simulation ont été comparés avec la méthode de partition des coordonnées. Des résultats similaires ont été obtenus prouvant une acceptable stabilité et précision de la méthode utilisée pour résoudre ce problème.

Mots clés *: systèmes DAE, ODE, systèmes d'Euler-Lagrange, modélisation dynamique, systèmes multicorps.*

الملخص.

تناول المقال محاكاة النماذج الديناميكية لأنظمة متعددة الاجسام المقيدة التي يمكن أن تصاغ على أنها مجموعة من المعادلات التفاضلية الجبرية)م ت ج .(نحن نعتبر النموذج الحركي العام الناتج عن صياغة أويلر-لاغرانج . نحن ندرس الحل عن طريقة تخفيض المؤشر. لتوضيح تحليلنا نحن نستمد نموذجا من مثال آلية المنزلق -مُدوِّرة.. لقد حصلنا على نموذج ديناميكي استندنا الى اثنين من المتغيرات التي تقدم أسهل التحليل والتنفيذ. تمكنا من حل المشكلة) م ت ج (بواسطة برنامج (ode5s) لماتلاب وهو مخصص لحل المعادلات التفاضلية العادية الشديدة. وقد تم مقارنة نتائج المحاكاة عن طريقة تقسيم. الإحداثيات. و تم الحصول على نتائج مماثلة تثبت الاستقرار ودقة الطريقة المستخدمة لحل هذه المشكلة.

الكلمات المفتاحية: . ، النمذجة الديناميكية ، الأنظمة متعددة الاجسام أويلر لاغرانج، أنظمة, ODE، DAE.

Introduction :

Constrained mechanical multibody systems can be found in various scientific and technologic applications such as robotics, biomechanics of locomotion, vehicle engines, and machinery [1, 2, 3,4]. Some well established methods are available for modeling constrained mechanical multibody systems such as Newton-Euler Law, Hamilton principle, and Lagrange multiplier, etc [1,2].

From the mathematical point of view, the dynamic models of constrained multibody systems fall into two main categories: Differential Algebraic Equations (DAEs) and Ordinary Differential Equations (ODEs). The difference between ODEs and DAEs comes from the choice of the descriptive parameters used and the topology of the mechanisms. In particular, DAE systems are subject to geometric and physical constraints in their mechanisms. [1, 2].

Several techniques have been proposed to solve DAE systems [5,6, 7]. An obvious approach consists of differentiating the constraints one or more times with respect to time. Then to replace the geometric constraints by their derivatives to convert the DAE problem into a mathematically equivalent ODE problem before applying some well known numerical integration methods. This approach corresponds to the index-reduction method. However, the main issue with this technique is that the numerical solution of the system may not satisfy the constraints of the original DAE problem due to error propagation known as drift-off phenomena. This means that, in general, the pure mathematical equivalence between the DAE and ODE problems is sensitive and not necessarily preserved by the computational procedures used for their solution [7,8].

According to the literature [8, 9], there are two main types of methods to solve this problem. The first class methods aims to reducing the system description to a minimum number of coordinates by finding a set of independent coordinates. These are the projection methods. The system is described in state-space form. Among these methods, we can cite full reduction of the system to a purely ODE form, which can be obtained by means of the Coordinate Partitioning. The second class of methods consists of index reduction of the original problem. It introduce additional unknowns leading to augment the original system and then to apply stabilization techniques.

In this paper, we consider the general dynamical model of the multibody systems obtained by means of Euler-Lagrange formulation as a DAE problem. We illustrate our analysis and simulation by the example of the slider-crank mechanism. By manipulating the constraints equations, we derive a mathematically equivalent DAE model with two variables. We solve the DAE problem by using a Matlab function dedicated to stiff ODEs systems (ode13s). Comparisons of our simulation results with other techniques will be presented.

2 - The General formulation of the dynamic multibody systems:

The Lagrange formulation is suitable for modeling constrained mechanical multibody systems. The modeling requires to define some parameters which are used for the system representation. These parameters are coordinates which enable to describe the positioning and the movement of the system.

2.1-Dynamical Model:

The equations of motion given by Euler-Lagrange formulation is a set of differential equations of a second order associated to a set of geometrical constraint equations. They are often expressed in the following form of a DAE problem [1,2, 8, 9]:

$$M(t,q,\overset{*}{q}).\overset{**}{q} = -\Phi_q^T.\lambda + F(t,q,\overset{*}{q}) \quad (1)$$

$$\Phi(t,q) = 0 \quad (2)$$

Here, q is a vector of generalized co-ordinates, $\dot{q}$ is a vector of generalized velocities. M(t, q, $\dot{q}$) is the mass matrix du system de dimension (nxn), Φ is a vector of the constraint equations and λ is a vector of Lagrange multipliers [1, 2]. Φ_q is the Jacobian matrix of constraints. Φ_q^T is the transpose of the Jacobian matrix of constraints. F(t, q, $\dot{q}$) the vector of the generalized forces (other than the constraint forces).

To determine uniquely a solution to this problem, it is necessary to add initial conditions which are associated to the set of differential equations: q(t0)=q0 and $\dot{q}$(t0)= $\dot{q}$0. These initial conditions have to satisfy the consistency of the constraints and their derivatives at any instant of time.

2.2- State Space representation:

Another common equivalent representation of the previous DAE problem uses the state space variables (position coordinates p and velocities v). The DAE problem can be written in the following form :

$$\begin{cases} \dfrac{dp}{dt} = v \\ M.\dfrac{dv}{dt} = F - \Phi_q^T.\lambda \\ \Phi(t,p,v) = 0 \end{cases} \quad (3)$$

with initial conditions: p (0) = p_0, v (0) = v_0. The given vectors p_0 and v_0, which specify the initial configuration and initial velocity, are chosen so satisfy the consistency of the constraint equations and their derivatives.

$$\Phi(p_0) = 0$$

$$\Phi_q(p_0).v_0 = 0$$

2.3- Augmented DAE Representation

The DAE systems are characterized by their differentiation index which is defined as the number of differentiation of the constraints in order to transform the DAE problem into a mathematical equivalent ODE one. In this case, the problem is index-3 [1-2, 6-9].

Reducing the index by deriving twice the constraint equations, leads to transform the DAE problem from index-3 to index-1 as follows.

Considering the position constraint equations:

$$\Phi(\boldsymbol{q}, t) = 0$$

by deriving once, we get the velocity constraint equations: $\Phi_q.\dot{q} = 0$ (4)

by deriving twice, we obtain the acceleration constraint equations:

$$\Phi_q.\ddot{q} = -(\Phi_q.\dot{q})_q.\dot{q} - 2.\Phi_q.\dot{q} - \Phi_{tt} = \gamma \tag{5}$$

By coupling the equations of motion (1) with the acceleration constraint equations (5), we can obtain the following DAE system:

$$\begin{bmatrix} M & \Phi_q^T \\ \Phi_q & 0 \end{bmatrix} \begin{bmatrix} \ddot{q} \\ \lambda \end{bmatrix} = \begin{bmatrix} F \\ \gamma \end{bmatrix} \tag{6}$$

Assuming that the Jacobian $\mathbf{\Phi_q}$ has full row rank, it can be proved that because the kinetic energy of a system is always positive, the coefficient matrix of the linear system above is nonsingular [6-9]. This means that a solution exists and is unique. We have a set of *n* generalized coordinates, they need to satisfy the set of *m* constraints present in the system:

$$\Phi(\mathbf{q}, t0) = 0$$

This augmented system (6) is equivalent to Lagrange's equations (1), if and only if the initial conditions of the problem satisfy the constraint conditions.

3-Dynamic Modeling of a Slider-Crank System

3.1- Variables selection and Geometric Constraints:

The graphical representation of the two-dimensional slider-crank mechanism is given in Fig.1. It is constituted of two mobile bodies: the crank (length l_1 , mass m_1, inertia J_1) and the coupler (length l_2. mass m_2 , Inertia J_2). The slider has a mass *m*. To drive the system, the external effort (*Mom)* is exerted at the base of the crank element (point A), the crank rotates leading the slider to move left to right in the x-direction. We will be interested by the motion of the slider.

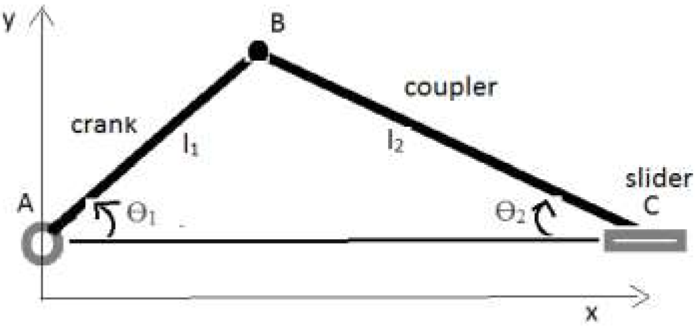

Fig.1: Slider-crank mechanism

To describe the topology and the dynamic of the system, we can use absolute coordinates or relative coordinates which are joints. This selection affects the number and the nature of equations. In principle, we only need one coordinate such as θ_1, but since there is not an obvious connection between θ_1 and the complete configuration of the mechanism, we introduce other coordinates.

Generally, to establish the constraint equations, a possible choice is the three variables: the angles Θ_1, Θ_2 and the horizontal displacement of the slider *x*. Considering the triangle ABC, we can get the following two constraint equations.

$$\Phi = \begin{bmatrix} l_1.\sin\theta_1 - l_2.\sin\theta_2 = 0 \\ l_1\cos\theta_1 + l_2.\sin\theta_2 - x = 0 \end{bmatrix} \quad \begin{matrix} (7) \\ (8) \end{matrix}$$

We notice that we have two independent variables (Θ_1, x) and one dependant variable Θ_2. Since these two equations are redundant, then to simplify the problem formulation, we only use the first constraint equation, the second with be deduced once Θ_1 and Θ_2 are determined.

3.2- Dynamic Model of the Slider-Crank system:

According to the Euler-Lagrange formulation, the dynamical model of the two dimensional slider crank mechanism can be formulated as a DAE problem of the general following form (1) et (2):

$$M(t, q, \dot{q}).\ddot{q} + \Phi_q{}^T \lambda = F(t, q, \dot{q})$$

$$\Phi(t, q) = 0$$

To describe the system, many possible choice of variables can be selected [7-12]. As in [10], we have selected two variables Θ_1 and Θ_2. This choice enable to decouple the equations and facilitates expressions of the constraint derivatives. We have reduced the system from index-3 to index-1 by deriving twice the constraint. The obtained expressions of $M(t,q,\overset{*}{q})$ and $F(t,q,\overset{*}{q})$ are :

$$M(\theta_1,\theta_2)=\begin{bmatrix} l_1^2.(\frac{1}{4}m_1+m_2+m_3)+J_1 & -l_1.l_2.\cos(\theta1+\theta2).(\frac{1}{2}m_2+m_3) \\ -l_1.l_2.\cos(\theta1+\theta2).(\frac{1}{2}m_2+m_3) & l_2^2.(\frac{1}{4}m_2+m_3)+J_2 \end{bmatrix} \quad (9)$$

$$F(\theta_1,\theta_2,\overset{*}{\theta}_1,\overset{*}{\theta}_2)=\begin{vmatrix} -l1.g.(\frac{1}{2}m1+m2+m3).\cos(\theta1)-l1l2.\overset{*2}{\theta}_2.\sin(\theta1+\theta2).(\frac{1}{2}m2+m3)+Mom \\ l2.g.(\frac{1}{2}m2+m3).\cos(\theta2)-l1l2.\overset{*2}{\theta}_1.\sin(\theta1+\theta_2).(\frac{1}{2}m2+m3) \end{vmatrix} \quad (10)$$

Let's reconsider the first constraint equation (7). By deriving this constraint once w.r.t, we get the expression :

$$\Phi_q\overset{*}{q}=l1.\cos\theta_1.\overset{*}{\theta}_1-l2.\cos\overset{*}{\theta}_2=\begin{bmatrix} l1\cos\theta1 & -l2\cos\theta2\end{bmatrix}\begin{bmatrix}\overset{*}{\theta}1\\ \overset{*}{\theta}2\end{bmatrix}=0 \quad (11)$$

from which, we obtain the Jacobian:

$$\Phi_q=\begin{bmatrix} l_1\cos\theta_1 & -l_2\cos\theta_2\end{bmatrix} \quad (12)$$

and its transpose:
$$\Phi_q^T=\begin{bmatrix} l_1.\cos\theta_1 \\ -l_2.\cos\theta_2\end{bmatrix}. \quad (13)$$

To bring the problem into index-1, we should derive twice the constraint equation. It gives:

$$\Phi_q.\overset{**}{q}=\gamma=-l1.\sin\theta1.\overset{*}{\theta}_1^2+l1.\cos\theta.\overset{**}{\theta}1+l2.\sin\theta_2.\overset{*}{\theta}_2^2-l2.\cos\theta_2.\overset{**}{\theta}_2 \quad (14)$$

The expressions (9), (10), (12), (13) and (14) are used in order to obtain the augmented DAE form (6).
We need also to add the initial conditions:

$$\theta_1(0)=\theta_{10},\ \overset{*}{\theta}1(0)=\overset{*}{\theta}_{10},$$

$$\theta_2(0)=\theta_{20}.\theta_1(0)=\theta_{10},\ \overset{*}{\theta}2(0)=\overset{*}{\theta}_{20}.$$

4-Numerical Solution and simulation:

4.1-Numerical solution:

To solve the system (1) and (2) requires to predict the motion of the system (q(t); q˙(t)), from an initial configuration (q(t = 0); q˙(t = 0)), by time-integrating the accelerations q¨(t). The augmented form (6) corresponds to a DAE index-1 problem. Since this DAE system contains time derivatives of second order, it has to be reformulated to obtain the form compatible with ODE like systems [14-16]:

$$V(t,y).\overset{*}{y}=f(t,y) \quad (15)$$

with initial conditions: $y(t_0)=y_0$

The resolution technique requires to augment the system by adding some other variables and equations.

If we note: $x=\begin{pmatrix}\theta1\\ \theta2\end{pmatrix}$

and define the state vector as:

$$y=\begin{pmatrix} x\\ \overset{*}{x}\\ \overset{**}{x}\\ \lambda\end{pmatrix}=\begin{pmatrix}\theta_1 & \theta_2 & \overset{*}{\theta}_1 & \overset{*}{\theta}_2 & \overset{**}{\theta}_1 & \overset{**}{\theta}_2 & \lambda\end{pmatrix}^T$$

then, the augmented problem can be expressed in the form:

$$\begin{bmatrix} E & 0 & 0 & 0\\ 0 & E & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}.\begin{pmatrix}\overset{*}{x}\\ \overset{**}{x}\\ \overset{***}{x}\\ \overset{*}{\lambda}\end{pmatrix}=\begin{pmatrix}\overset{*}{x}\\ \overset{**}{x}\\ M.\overset{**}{x}-Q_q^T.\lambda-F\\ \overset{**}{g}\end{pmatrix} \quad (16)$$

where V is a diagonal but singular matrix and E is a unit (2*2) matrix. under this form, some implemented codes have been dedicated to solve this problem provided giving consistent initial conditions.

$$y=\begin{pmatrix}\theta_{10} & \theta_{20} & \overset{*}{\theta}_{10} & \overset{*}{\theta}_{20} & \overset{**}{\theta}_{10} & \overset{**}{\theta}_{20} & \lambda_0\end{pmatrix}^T \quad (17)$$

To solve this problem, we have used the Matlab function: ***ode15s*** [12,14, 15]. It numerically integrates the system (15) which is expressed as (16-17) from an initial time t_0 to a final time t_f. It has many advantages over *ode45*. It is recommended in case of stiff ODE and DAE problems. It can solve problems in form (16) with a *V(t,y)* matrix that is singular which is our case. It can check and adapt the consistency of the initial conditions.

From the algorithmic point of view, ode15s is a variable-order solver based on the Numerical Differentiation Formulas (NDFs) [4]. Optionally, it uses the backward differentiation formulas (BDFs, also known as Gear's method) that are usually less efficient. ode15s is a multistep solver [13-15].

4.2-- Simulation and Graphical Results :

The simulation has been performed with the ode15s. It has applied to the double pendulum and to the slider crank mechanism. The obtained results have been in concordance with those obtained in the literature. In this section, we present in Fig.2 the graphical results concerning the slider-crank mechanism: the temporal evolutions of the angle $\Theta_1(t)$ and of the slider displacement x(t). We notice that x(t) has been computed a posteriori after calculating $\Theta_1(t)$ and $\Theta_2(t)$ from the constraint equation (8) as : $x = l_1.\cos\theta_1 + l_2.\sin\theta_2$. This a posteriori checking has ensured the respect of the constraint equations during the calculations and thus ensuring the stability and the accuracy of the solver for these type of problems.

The chosen parameters are: l_1=0.2; l_2= 1; m_1=1; m_2=1; J_1 = 1; J_2=1; m=1; Mom= 10; g= 9.81.

The vector of initial conditions is: y0 = [0.78 asin((l_1/l_2)*sin(0.78)) 0 0 0 0 0]^T. In our implementation, the initial value of Θ_{10} is given, the second one is computed automatically by means of the first constraint (7) as Θ_{20} = asin((l_1/l_2)*sin(Θ_{10})) to ensure the consistency of the constraint equations. The matrix V is singular and its diagonal is: diag(V) = [1,1,1,1,0,0,0];

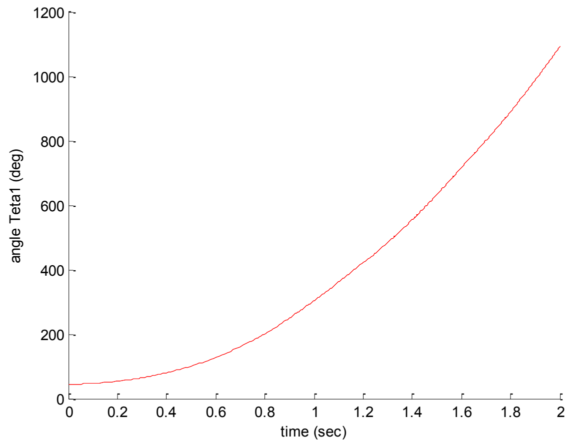

Fig.2: Temporal evolution of the angle $\Theta_1(t)$

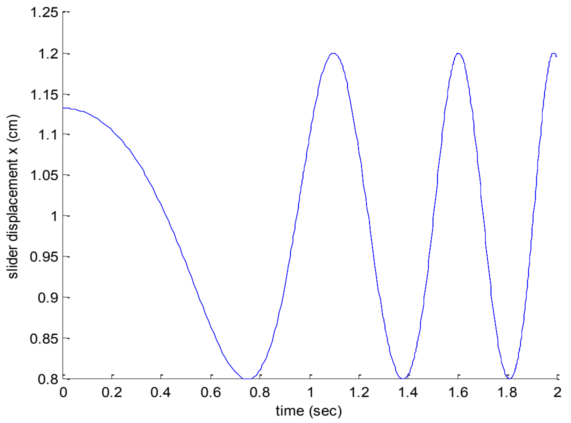

Fig.3: Temporal evolution of the slider displacement x(t).

The obtained results are similar to those obtained by the partitioning coordinates method which uses conventional ODE systems and which has been applied to the same slider-crank mechanism [4]. Many other concordant results have been also noticed with other approaches applied to the slider-crank mechanisms [6-11]. A comparison of different numerical approaches for solving the problem of the slider-crank mechanism is reported in [16].

The simulation results and the concordance of the applied technique with the other techniques proving to conclude that the method is stable and accurate enough.

5-Conclusion

The paper has dealt with the analysis and simulation of some dynamical models of constrained multibody systems which have been formulated as a set of Differential Algebraic Equations (DAEs). We have investigate the solution with the index reduction method. We have illustrated our analysis by the example of the slider-crank mechanism. We have derived a dynamical model based on two variables offering an easier analysis and implementation. We have successfully solved the DAE problem with the Matlab function (ode15s) which is dedicated to solve stiff Ordinary Differential Equations (ODEs).

Concordant simulation results have been obtained in comparison to other methods such as the partitioning coordinates method proving the stability and the accuracy of the used DAE method. The choice of the variables for a particular derived model can influence the choice of the DAE resolution method and also the complexity and computation cost.

These mathematical results have also enabled coherent physical interpretations of the functioning of the slider-crank mechanisms.

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