PERFORMANCE UTILITY OF MULTI-STATE CONSECUTIVE k -OUT-OF- $n:G$ SYSTEMS

Reçu le 29/05/2006 – Accepté le 23/12/2008

Résumé

L'importance en fiabilité d'un composant est une mesure quantitative de la contribution d'un composant à la fiabilité du système. Dans ce papier, on discute la contribution individuelle d'un composant à la performance en utilité d'un système k - consécutifs parmi- $n:G$ à plusieurs états, basée sur l'importance en utilité d'un état d'un composant dans un système à états multiples, introduite par S.Wu et L.Y.Chan. Un exemple illustrant ceci est traité.

Mots clés : Fiabilité, importance de l'utilité du composant, multi-états, systèmes k consécutifs parmi- $n:G$.

Abstract

Reliability importance of a component is a quantitative measure of the importance of the individual component in contributing to system reliability. In this paper, we discuss the contribution of an individual component to the performance utility of a multi-state consecutive k -out-of- $n:G$ system, based on the utility importance of a component's state in multi-state systems given by S.Wu and L.Y.Chan (2003), and we illustrate by an example.

Keywords: Reliability, component importance utility, multi-state, consecutive k -out-of- $n:G$ systems.

Classification AMS : 90B25, 60K10.

S. BELALOUI

Département de Mathématiques,
Faculté des Sciences Exactes
Université Mentouri,
Constantine.
Algérie

أهمية إمكانية الاشتغال لعنصر هي قياس كمي لإسهام العنصر في إمكانية اشتغال النظام. في هذا العمل، متعدد الحالات، و هذه n متوالية من بين k نناقش المساهمة الفردية لعنصر إلى منفعة تجلية نظام المناقشة على أساس منفعة الأهمية لحالة عنصر متعدد الحالات في نظام متعدد الحالات. نستعمل التعريف المعطى من طرف S.Wu, L.Y Chan

الكلمات المفتاحية: إمكانية الاشتغال، منفعة أهمية مركب، متعدد الحالات، نظام n متوالية من بين k

I. INTRODUCTION

Acronyms:
M.S: multi-state.
iff : if and only if.

A system consists of many components performing various functions. One of the most important measures of the performance of a system is its reliability. The reliability of a system is defined to be the probability that the system will perform its functions satisfactorily for a certain time period under specified conditions. To achieve high reliability for a complex system, it is necessary to identify the components that have the greatest effect on the system reliability. Such items can be identified using importance measures. So, importance measures are important tools to evaluate and rank the impact of individual components within a system. The reliability importance of a component is the rate at which system reliability improves as the component reliability improves. This information can be used to determine which components should be improved first in order to make the largest improvement in system reliability. Extensive research [1]-[4] for importance measures is available for binary systems. Birnbaum-importance measures the contribution of component-reliability to the system reliability [4]-[5] Structural-importance measures the topographic importance of a position in the system [6]-[7] Criticality-importance corresponds to the conditional probability of failure of a component, given that the system has failed [4]-[5] Joint-importance measures how components in a system interact and contribute to the system reliability [8]-[9] Traditional reliability theory has been on binary applications. In the binary system: the system and its components are allowed to have only two possible states (completed failure and perfect functioning). In the M.S system: both the system and its components may experience more than two states, for example, completely failed, partially functioning and perfect functioning. There are numerous examples of M.S systems, with than 2 ordered or unordered states at the system level, or the component level. As water distribution, a power plant which has states 0,1,2,3,4 that correspond to generating electricity of 0 %, 25 %, 50 %, 75% , 100 % of its full capacity is an example of a M.S system that has ordered multiple states [10] . A nuclear reactor system or a pumping system, telecommunications, a light-emission diode which emits red, green, and yellow lights under different inputs. Furthermore, a state in a system may take a continuous range of quantitative measurement instead of discrete levels, for example, a branking system might produce an output branking force ranging from 250 to 300 kilograms. Generally, the elements of these systems degrade gradually, reducing their capacity, and the overall capability of the system. The definition of reliability as given under the binary assumption is no longer valid in the M.S context. Different measures of system performance are warranted. In recent years, M.S system reliability analysis has received considerable attention. Researches have realized that for some systems, erroneous appraisal of system reliability could lead to :

1) incorrect system modelling.
2) incorrect system reliability computation. And /or
3) incorrect conjectures regarding reliability dependent measures.

Theoretical and applied studies have been devoted to the areas of M.S system reliability, simulation, approximation methodologies, and optimization [11]. Some extension of importance measures from binary systems to M.S systems has been extensively investigated. El-Neweihi, et al. [12] analysed the theoretical relationships between M.S system reliability behavior, and M.S component performance. Barlow and Wu [13] characterize component state criticality as a measure of how a particular component state affects a specific system state.Griffith [14] formalized the concept of M.S system performance, and studied the impact of component improvement on the overall system reliability behavior. Moreover, Griffith introduced the concept of reliability importance vector for each system component. Through this concept, a generalization of the binary Birnbaum importance measure can be extented to M.S case. Levitin and Lisnianski [15] proposed importance and sensitivity measures for M.S systems with binary capacitated components. Importance measures are obtained through the universal generating function. Zio and Podofillini [16] present M.S extention for Reliability Achievement Worth (RAW), Reliability Reduction Worth (RRW), Fussell-Vesely Importance (FV), and Birnbaum for M.S systems. Their results pertain to the importance of individual components state levels. Monte-Carlo simulation methods are used to imitate the stochastic nature of the M.S components, and generate the proposed importance measures. J.R.Marquez and D.V.Coit [17] present and evaluate composite importance measures for M.S systems. They present (type 1) importance measures that are involved in measuring how a specific component affects M.S system reliability, and (type 2) importance measures have focused on investigating how a particular component state or a set of states affects M.S system reliability. Few publications discuss how the particular states of a component contribute to a M.S system, and how the presence of a component and a particular state of a component affect the contributions of other components in the system. Such an investigation has theoretical importance as well as practical value, because the knowledge gained enables efficient design of the system. In a binary system, reliability optimization mainly deals with maximizing the system reliability under constrains such as cost, weight, and /or size, or on minimizing the cost under reliability constrains. This optimization task is by no means trivial, unless the system is very simple. In M.S systems where components have more than 2 states and the performance utility of the system is to be maximized, the optimization task is obviously more difficult. S.Wu and L.Y.Chan [18] introduced a new utility importance for measuring the contribution of various components states to the system-performance, and compared this utility importance to Griffith's importance. So research efforts have been focused on generalizing frequently used binary importance measures to accommodate the M.S behavior. These approaches characterize, for a given component, the most important state with regard to its impact on system reliability.

In this paper, The consecutive k -out-of - n systems are investigated, because they have a wide range of applications, as telecommunications, pipeline....By using the performance utility-function and the component importance utility, we focus on how a specific component and a particular state or a set of states affects M.S system reliability. First, we present the formula which computes the distribution state of M.S consecutive k -out-of- $n:G$ systems [19] , and we calculate the performance utility of these systems. We specify the component and the state which contributes the most. An example is given to illustrate this concept.

2 Notations and Nomenclature:

n : number of components in the system.

S: {0,1,…,M} , M the perfect functioning, 0 : the complete failure.

j : an integer, 0

a_j : system utility level when the system is in state j.

$$0 \le a_0 \le a_1 \le .... \le a_M$$

X_i : a random variable which represents the state of component i in the system.

X : $(X_1, X_2,..,.X_n)$: vector of components states.

$\Phi(X)$: system-state structure function: $\Phi(X) \in S$.

k_j : minimum number of consecutive components with $X_i \ge j, i \in \{1,2,3,....,n\}$.

$p_{i,j} = P\{X_i = j\}$.

$R_{s,j} = P\{\Phi(X) = j\}$.

$U = \sum_{j=0}^{M} a_j P\{\Phi(X) = j\}$, performance utility-function of a system.

$I^G(i)$: Griffith importance vector of component i .

$I^{UI}(i)$: utility importance of component i.

- M.S minimal path vector : $Y \in S^n$ is a minimal path vector to system-state level j iff

$$\Phi(Y) \ge j \text{ and } \Phi(X) < j \text{ for all } X < Y$$

- M.S minimal cut vector : $Y \in S^n$ is a minimal cut vector to system-state level j iff

$$\Phi(Y) < j \text{ and } \Phi(X) \ge j \text{ for all } X > Y$$

Let there be 2 component state vectors X, Y then

$$X < Y \text{ if } x_i \le y_i \text{ for all } i, \text{ and } x_i < y_i \text{ for at least one } i$$
$$X > Y \text{ if } x_i \ge y_i \text{ for all } i, \text{ and } x_i > y_i \text{ for at least one } i$$

* The utility of the system when it is in state j is represented by a_j . It represents the net profit or loss the system can generate if it is in state j .

3.1 Assumptions:

1) The system is M.S monotone
   - $\Phi(X)$ is nondecreasing in each argument.
   - $\Phi(j, j,..., j) = j$ for $j \in S$.

2) The X_i are mutually s-independent.

3) The system and each component has a zero state and M nonzero states.

4) The possible states of each component and of the system are ordered:

$$state0 \le state1 \le ...... \le stateM.$$

The first assumption roughly says that improving one of the components can not harm the system.

3 Griffith importance:

Griffith proposed the importance vector to study component i $(i = 1,2,....,n)$ in a M.S system :

$$I^G(i) = \left(I_1^G(i), I_2^G(i), ..., I_M^G(i)\right),$$

$$I_m^G(i) = \sum_{j=1}^{M}\left(a_j - a_{j-1}\right)\left[P\{\Phi(m_i, X) \geq j\} - P\{\Phi((m-1)_i, X) \geq j\}\right]$$

$m = 1, 2,..., M$

The $I_m^G(i)$ in Griffith's importance vector can be interpreted as the change of the system performance when component i deteriorates from state m to state $m-1$. A drawback of $I^G(i)$ is that it measures only how the change of particular component affects the system performance, but does not measure which component affects it the most, or which state of a certain component contributes the most. However, the extent to which a component and its states affect the system is a major concern to the system designer and the system controller. So, Wu and Chan [18] introduced a new performance utility importance function as follows:

$$\begin{aligned} I_m^{UI}(i) &= \sum_{j=0}^{M} a_j P\{\Phi(X) = j, X_i = m\} \\ &= \sum_{j=0}^{M} a_j P\{\Phi(X) = j / X_i = m\}.P\{X_i = m\} \\ &= \sum_{j=0}^{M} a_j P\{\Phi(X) = j / X_i = m\} p_{i,m} \\ &= p_{i,m} \sum_{j=0}^{M} a_j P\{\Phi(m_i, X) = j\} \end{aligned}$$

$I_m^{UI}(i)$ can be interpreted as the contribution of state m of component i to the system. And the utility importance of component i can be defined as the vector:

$$I^{UI}(i) = \left(I_0^{UI}(i), I_1^{UI}(i), ..., I_M^{UI}(i)\right)$$

A relationship between $I_m^{UI}(i)$ and coordinate $I_m^G(i)$ is given by:

$$\frac{\partial I_m^{UI}(i)}{\partial p_{i,m}} - \frac{\partial I_{m-1}^{UI}(i)}{\partial p_{i,m-1}} = I_m^G(i), \qquad m = 1,2,...,M$$

And the performance utility function U can be expressed in terms of $I_m^{UI}(i)$:

$$\begin{aligned} U &= \sum_{j=0}^{M} a_j P\{\Phi(X) = j\} \\ &= \sum_{j=0}^{M} a_j \left[\sum_{m=0}^{M} .P\{\Phi(X) = j / X_i = m\} P\{X_i = m\}\right] \\ &= \sum_{m=0}^{M}\sum_{j=0}^{M} a_j P\{\Phi(m_i, X) = j\} p_{i,m} = \sum_{m=0}^{M} I_m^{UI}(i) \end{aligned}$$

The equation above shows that a state m of component i with larger $I_m^{UI}(i)$ contributes appreciably more to the system performance utility.

4 The M.S consecutive k -out-of- $n : G$ system:

A M.S consecutive k -out-of- $n : G$ system is a system with n linearly arranged components, which are labelled $1,2,..,n.$ The system works iff at least k consecutive components work.
We consider that:

$$\left\{\begin{array}{l} X_i \geq j \Rightarrow \text{the component works} \\ X_i < j \Rightarrow \text{the component fails} \end{array} \quad i = 1,2,...,n \quad j = 1,2,...,M\right\}$$

and similarly for the system:

$$\left\{\begin{array}{l} \Phi(X) \geq j \Rightarrow \text{the system works} \\ \Phi(X) < j \Rightarrow \text{the system fails} \end{array}\right\}$$

Because j can have various values, the terms "working" and "failure" have dynamic meanings.

The system-state structure function Φ is given by:

$$\Phi(X) = \max_{1 \leq i \leq n-k+1} \ \min_{i \leq l \leq i+k-1} X_l$$

The reliability of the system is given by :

$$P\{\Phi(X) \geq j\} = \sum_{\alpha=j}^{M} P\{\Phi(X) = \alpha\} = \sum_{\alpha=j}^{M} R_{s,\alpha} \ , \quad j = 1,2,...,M$$

In [19],[20] k is supposed not constant but it varies with j i.e one define the M.S consecutive k -out-of- $n:G$ system where in maintaining at least a certain system-state level might require a different number of consecutive components to be at a certain state or above (the required number of consecutive components depends on the system state level) so:

Definition [20] :

$\Phi(X)\geq j$ iff at least k_l consecutive components are in state l or above for all l $(1\leq l\leq j, \quad j=1,...,M)$.

The condition in this definition can also be phrased as follows: $\Phi(X)\geq j(j=1,2,...,M)$ if at least k_j consecutive components are in state j or above; at least k_{j-1} consecutive components are in state $j-1$ or above; ...., and at least k_1 consecutive components are in state 1 or above.

The system state distribution is expressed as [19] :

$$R_{s,j}=P\{\Phi(X)=j\}=\sum_{k=k_j}^{n}\left[R_j(k,n)+\sum_{h=j+1,k_h>1}^{M}H_k^j(h)\right]$$

where $R_j(k,n)$ is the probability that exactly k components are in state j , which include among them at least k_j consecutive components, and the other $n-k$ components are below j and $H_k^j(h)$ is the probability that:

- at least 1 and at most k_h-1 components are in state h $(h>j)$.
- at most k_u-1 components are in state u for $j<u<h.$
- the total number of components $\geq j$ is k , which include among them at least k_j consecutive components.
- $n-k$ components are at states below j.

So, $H_k^j(h)$ is calculated as follows :

$$H_k^j(h)=\sum_{i_1=1}^{k_h-1}\sum_{i_2=0}^{k_{h-1}-1-I_1}\sum_{i_3=0}^{k_{h-2}-1-I_2}\cdots\cdots\sum_{i_{h-j}=0}^{k_{j+1}-1-I_{h-j-1}} R\left[\left(h^{i_1},(h-1)^{i_2},....,(j+1)^{i_{h-j}},j^{k-I_{h-j}}\right),n\right]$$

where:

$$I_l=\sum_{\alpha=1}^{l}i_\alpha \qquad ;l=1,2,...,h-j$$

and $R\left[\left(h^{i_1},(h-1)^{i_2},....,(j+1)^{i_{h-j}},j^{k-I_{h-j}}\right),n\right]$ is the probability that there are exactly:

- i_1 components at level h.
- i_2 components at level $h-1$.
- .....
- i_{h-j} components at level $j+1$.
- $k-I_{h-j}$ components at level j.
- the remaining $n-k$ components are at states below j.

5 Performance utility of the M.S consecutive k -out-of- $n:G$ system:

In this paper, we suppose that k is constant, i.e k is s-independent of the value of the system state level. In other words; in maintaining at least a certain system state level requires the same number of consecutive components to be at or above a certain state. And we compute the utility importance of a component in the M.S consecutive k -out-of- $n:G$ system, based on the definition given by [18] ,

5.1 Theorem:

The utility importance of state $m(0\leq m\leq M)$ of component i in the system is given by:

$$I_m^{UI}(i)=p_{i,m}\left\{\begin{array}{l}\sum_{j=0}^{m-1}a_j\Delta_{n,m,m}^j+a_m\left[L_{n,m}^m+\Delta_{n,m,m+1}^*\right]\\ +\sum_{j=m+1}^{M}a_j\left[L_{n-1,m}^j+\Delta_{n-1,m,j+1}^j\right]\end{array}\right\} \quad \text{if} \quad 0<m<M$$

$$=p_{i,M}\left\{\sum_{j=0}^{M-1}a_j\Delta_{n,M,M}^j+a_M L_{n,M}^M\right\} \quad \text{if} \quad m=M$$

$$=p_{i,0}\left\{a_0\left[L_{n,0}^0+\Delta_{n,0,1}^*\right]+\sum_{j=1}^{M}a_j\left[L_{n-1,0}^j+\Delta_{n-1,0,j+1}^j\right]\right\} \text{ if } \quad m=0$$

Proof:
The distribution state of the system is :

$$P\{\Phi(X)=j\}=\sum_{k'=k}^{n}\left[R_j\left(k',n\right)+\sum_{h=j+1}^{M}H_k^j(h)\right]$$

we can see that :

$$*\,m=j=M\Rightarrow P\{\Phi(m_i,X)=j\}=\sum_{k'=k}^{n}\left[R_j\left(m_i,k',n\right)\right]$$

$$*\,m=j<M\Rightarrow P\{\Phi(m_i,X)=j\}=\sum_{k'=k}^{n}\left[R_j\left(m_i,k',n\right)+\sum_{h=j+1}^{M}H_{k'}^{j}(m_i,h)\right]$$
$$=\sum_{k'=k}^{n}\left[R_j\left(m_i,k',n\right)+\sum_{h=m+1}^{M}H_{k'}^{j}(m_i,h)\right]$$

$$*\,m<j(j=M)\Rightarrow P\{\Phi(m_i,X)=j\}=\sum_{k'=k}^{n-1}\left[R_j\left(m_i,k',n\right)\right]$$

$$*\,m<j(j<M)\Rightarrow P\{\Phi(m_i,X)=j\}=$$
$$\sum_{k'=k}^{n-1}\left[R_j\left(m_i,k',n\right)+\sum_{h=j+1}^{M}H_{k'}^{j}(m_i,h)\right]$$

$$*\,m>j(j=M)\Rightarrow P\{\Phi(m_i,X)=j\}=0$$

$$*\,m>j(j<M)\Rightarrow P\{\Phi(m_i,X)=j\}=\sum_{k'=k}^{n}\left[\sum_{h=m}^{M}H_{k'}^{j}(m_i,h)\right]$$

The performance utility function of the system is :

$$U=\sum_{j=0}^{M}a_jP\{\Phi(X)=j\}=\sum_{m=0}^{M}I_m^{UI}(i)$$

and by definition:

$$I_m^{UI}(i)=p_{i,m}\sum_{j=0}^{M}a_jP\{\Phi(m_i,X)=j\}$$

the sum in $I_m^{UI}(i)$ can be written as :

$$\sum_{j=0}^{M}=\sum_{j=0}^{m-1}+\sum_{j=m}+\sum_{j=m+1}^{M}$$

the $\sum_{j=m}$ consists of a single term, then :

$$\sum_{j=0}^{M}a_jP\{\Phi(m_i,X)=j\}=\sum_{j=0}^{m-1}a_j\sum_{k'=k}^{n}\sum_{h=m}^{M}H_{k'}^{j}(m_i,h)+$$
$$a_m\sum_{k'=k}^{n}\left[R_j\left(m_i,k',n\right)+\sum_{h=m+1}^{M}H_{k'}^{m}(m_i,h)\right]+$$
$$\sum_{j=m+1}^{M}a_j\sum_{k'=k}^{n-1}\left[R_j\left(m_i,k',n\right)+\sum_{h=j+1}^{M}H_{k'}^{j}(m_i,h)\right]$$

if $0<m<M$

$$\sum_{j=0}^{M}a_jP\{\Phi(m_i,X)=j\}=\sum_{j=0}^{M-1}a_j\sum_{k'=k}^{n}H_{k'}^{j}(M_i,M)+a_M\sum_{k'=k}^{n}R_j\left(M_i,k',n\right)\quad \text{if } m=M$$

$$\sum_{j=0}^{M}a_jP\{\Phi(m_i,X)=j\}=a_0\sum_{k'=k}^{n}R_j\left(0_i,k',n\right)+\sum_{k'=k}^{n}\sum_{h=1}^{M}H_{k'}^{0}(0_i,h)$$
$$+\sum_{j=1}^{M}a_j\sum_{k'=k}^{n-1}\left[R_j\left(0_i,k',n\right)+\sum_{h=j+1}^{M}H_{k'}^{j}(0_i,h)\right]$$

if $m=0$

By setting :

$$\Delta_{s,w,z}^{j}=\sum_{k'=k}^{s}\sum_{h=j+1}^{M}H_{k'}^{j}(w_i,h);\qquad s=n-1,n\ ;\quad w=0,m$$

$$\Delta_{n,m,m}^{j}=\sum_{k'=k}^{n}\sum_{h=m}^{M}H_{k'}^{j}(m_i,h)\ ,\ \Delta_{n,M,M}^{j}=\sum_{k'=k}^{n}\sum_{h=M}^{M}H_{k'}^{j}(m_i,h)=$$
$$\sum_{k'=k}^{n}H_{k'}^{j}(M_i,M)$$

$$L_{s,w}^{j}=\sum_{k'=k}^{s}R_j\left(w_i,k',n\right);\qquad s=n-1,n\ ;\quad w=0,m,M$$

$$\Delta_{s,w,w+1}^{*}=\sum_{k'=k}^{s}\sum_{h=w+1}^{M}H_{k'}^{w}(w_i,h);\qquad s=n\ ;\quad w=0,m$$

it follows that :

$$I_m^{UI}(i) = p_{i,m}\sum_{j=0}^{M} a_j P\{\Phi(m_i, X) = j\}$$

$$= p_{i,m}\left\{\begin{array}{l} \sum_{j=0}^{m-1} a_j \Delta^j_{n,m,m} + a_m\left[L^m_{n,m} + \Delta^*_{n,m,m+1}\right] \\ + \sum_{j=m+1}^{M} a_j\left[L^j_{n-1,m} + \Delta^j_{n-1,m,j+1}\right] \\ \text{if } 0<m<M \end{array}\right\}$$

$$= p_{i,M}\left\{\sum_{j=0}^{M-1} a_j \Delta^j_{n,M,M} + a_M L^M_{n,M}\right\}$$

$$\text{if } m=M$$

$$= p_{i,0}\left\{a_0\left[L^0_{n,0} + \Delta^*_{n,0,1}\right] + \sum_{j=1}^{M} a_j\left[L^j_{n-1,0} + \Delta^j_{n-1,0,j+1}\right]\right\}$$

$$\text{if } m=0$$

Hence, we can obtain the performance utility :

$$U = \sum_{m=0}^{M} I_m^{UI}(i)$$

5.2 Remark:

We can consider the case where a component i may be in all the minimal paths sets of a M.S consecutive k -out-of- $n:G$ system, so the precedent theorem can be reformuled as follows :

5.3 Corollary:

The component i is in all the minimal paths sets of a M.S consecutive k -out-of- $n:G$ system. The utility importance of state $m(0 \le m \le M)$ of component i in the system is given by:

$$I_m^{UI}(i) = p_{i,m}\left\{\sum_{j=0}^{M} a_j \Delta^j_{n,m,m} + a_m\left[L^m_{n,m} + \Delta^*_{n,m,m+1}\right]\right\} \text{ if } 0<m<M$$

$$= p_{i,M}\left\{\sum_{j=0}^{M-1} a_j \Delta^j_{n,M,M} + a_M L^M_{n,M}\right\} \text{ if } m=M$$

$$= p_{i,0}\left\{a_0\left[L^0_{n,M} + \Delta^*_{n,0,1}\right]\right\} \quad \text{if } m=0$$

Proof :

We have

$$I_m^{UI}(i) = p_{i,m}\sum_{j=0}^{M} a_j P\{\Phi(m_i, X) = j\}$$

The sum in $I_m^{UI}(i)$ can be written as :

$$\sum_{j=0}^{M} = \sum_{j=0}^{M} + \sum_{j=m} + \sum_{j=m+1}^{M}$$

the last term in this sum is 0 because :

$$P\{\Phi(m_i, X) = j\} = 0 \ , \quad j = m+1, \dots, M$$

and we can see that :

$$* \, m = j = M \Rightarrow P[\Phi(m_i, X) = j = m] = \sum_{k'=k}^{n}\left[R_j(m_i, k', n)\right]$$

$$* \, m = j < M \Rightarrow P[\Phi(m_i, X) = j] = \sum_{k'=k}^{n}\left[R_j(m_i, k', n) + \sum_{h=m+1}^{M} H^m_{k'}(m_i, h)\right]$$

$$* \, m < j (j = M, j < M) \Rightarrow P[\Phi(m_i, X) = j] = 0$$

$$* \, m > j (j = M) \Rightarrow P[\Phi(m_i, X) = j] = 0$$

$$* \, m > j (j < M) \Rightarrow P[\Phi(m_i, X) = j] = \sum_{k'=k}^{n}\sum_{h=m}^{M} H^m_{k'}(m_i, h)$$

then :

$$I_m^{UI}(i) = \sum_{j=0}^{M} a_j P\{\Phi(m_i, X) = j\} = \sum_{j=0}^{m-1} a_j P\{\Phi(m_i, X) = j\} + \sum_{j=m} a_j P\{\Phi(m_i, X) = j\}$$

$$= \sum_{j=0}^{m-1} a_j \sum_{k'=k}^{n}\sum_{h=m}^{M} H^j_{k'}(m_i, h) + a_m \sum_{k'=k}^{n}\left[R_j(m_i, k', n) + \sum_{h=m+1}^{M} H^m_{k'}(m_i, h)\right]$$

$$\text{if } m<M$$

$$= \sum_{j=0}^{M-1} a_j \sum_{k'=k}^{n} H^j_{k'}(M_i, M) + a_M \sum_{k'=k}^{n}\left[R_j(M_i, k', n)\right]$$

$$\text{if } m=M$$

$$= a_0 \sum_{k'=k}^{n}\left[R_j(0_i, k', n) + \sum_{h=1}^{M} H^0_{k'}(0_i, h)\right]$$

$$\text{if } m=0$$

and by using the notations of the precedent theorem, one can have the result.

6.4 Example:

Let : a 4- component system with $k = 3$.

Both the system and the components can have 5 possibles states : 0 , 1 , 2 , 3 , 4 .

$$n = 4, \quad k = 3, \quad M = 4$$

and the components have the following probabilities to be in each state :

state / component	1	2	3	4
0	0.1	0.1	0.1	0.1
1	0.1	0.2	0.2	0.1
2	0.2	0.2	0.2	0.3
3	0.4	0.3	0.2	0.2
4	0.2	0.2	0.3	0.3

This system has two minimal path sets : $\{123\},\{234\}$, we can see that component 2 and component 3 are in the two minimal path sets.
For $(a_0, a_1, a_2, a_3, a_4) = (0,100,1000,2000,8000)$,
we comput the utility importance for component 2 which is in the two minimal path sets, and for component 1 which is not in all minimal path sets.
Case 1:

$$2 \in \{123\}, 2 \in \{234\}$$

By using the corollary, the utility importance for component 2 is as follows :
* At level: $m = M = 4$

$$I_4^{UI}(2) = p_{2,4}\left\{\sum_{j=0}^{3} a_j \Delta_{4,4,4}^j + a_4 L_{4,4}^4\right\}$$

$$\Delta_{4,4,4}^1 = \sum_{k=3}^{4} H_k^1(4_2,4) = H_3^1(4_2,4) + H_4^1(4_2,4)$$

$$H_3^1(4_2,4) = [(1410,0411)] + [(2410,1420,0421,0412)] + [(3410,1430,0413,0431)] + [(4410,1440,0441,0414)]$$

$$H_4^1(4_2,4) = [(1411)] + [(2411,1421,1412)] + [(3411,1431,1413)] + [(4411,1441,1414)]$$

$$\Delta_{4,4,4}^2 = H_3^2(4_2,4) + H_4^2(4_2,4)$$

$$H_3^2(4_2,4) = [(\times 422, 242\times)] + [(342\times, 243\times, \times 432, \times 423)] + [(442\times, 244\times, \times 424, \times 442)]$$

$\times$ can be 0 or 1

$$H_4^2(4_2,4) = [(2422)] + [(3422, 2432, 2423)] + [(4422, 2442, 2424)]$$

$$\Delta_{4,4,4}^3 = H_3^3(4_2,4) + H_4^3(4_2,4)$$

$$H_3^3(4_2,4) = [(343\times, \times 433)] + [(443\times, 344\times, \times 443, \times 434)]$$

$\times$ can be 0,1,2

$$H_4^3(4_2,4) = [(3433)] + [(4433, 3443, 3434)]$$

$$L_{4,4}^4 = \sum_{k=3}^{4} R_4(4_2,k,4) = R_4(4_2,3,4) + R_4(4_2,4,4)$$

$$R_4(4_2,3,4) = [(444\times, \times 444)], \quad R_4(4_2,4,4) = [(4444)]$$

$\times$ can be 0,1,2,3

then:

$$I_4^{UI}(2) = 71.532$$

* At level : $m = 3 < M = 4$

$$I_3^{UI}(2) = p_{2,3}\left\{\sum_{j=0}^{2} a_j \Delta_{4,3,3}^j + a_3\left[L_{4,3}^3 + \Delta_{4,3,4}^*\right]\right\}$$

$$\Delta_{4,3,3}^1 = H_3^1(3_2,3) + H_3^1(3_2,4) + H_4^1(3_2,3) + H_4^1(3_2,4)$$

$$H_3^1(3_2,3) = [(0311,1310)] + [(0321,0312,1320,2310)] + [(3310,1330,0331,0313)]$$

$$H_4^1(3_2,3) = [(1311)] + [(2311,1321,1312)] + [(3311,1331,1313)]$$

$$H_3^1(3_2,4) = [(4310,1340,0314,0341)], \quad H_4^1(3_2,4) = [(4311,1341,1314)]$$

$$\Delta_{4,3,3}^2 = H_3^2(3_2,3) + H_3^2(3_2,4) + H_4^2(3_2,3) + H_4^2(3_2,4)$$

$$H_3^2(3_2,3) = [(\times 322, 232\times)] + [(\times 332, \times 323, 332\times, 233\times)]$$

$\times$ can be 0,1

$$H_3^2(3_2,4) = [(432\times, 234\times, \times 324, \times 342)] \text{ , } \times \text{ can be } 0,1$$

$$H_4^2(3_2,3) = [(2322)] + [(3322, 2332, 2323)], \quad H_4^2(3_2,4) = [(4322, 2342, 2324)]$$

$$L_{4,3}^3 = [(333\times, \times 333)] + [(3333)] \quad \times \text{ can be } 0,1,2$$

$$\Delta_{4,3,4}^* = H_3^3(3_2,4) + H_4^3(3_2,4)$$

$$H_3^3(3_2,4) = [(\times 334, \times 343, 334\times, 433\times)] + [(\times 344, 434\times)] \quad \times \text{ can be } 0,1,2$$

$$H_4^3(3_2,4) = [(3334, 3343, 4333)] + [(3344, 4343, 4334)]$$

then:

$$I_3^{UI}(2) = 52.626$$

* At level : $m = 0$

$$I_0^{UI}(2) = p_{2,0}\{a_0[L_{4,0}^0 + \Delta_{4,0,1}^*]\}$$

$$I_0^{UI}(2) = 0$$

one can see that :

$$I_4^{UI}(2) > I_3^{UI}(2) > I_0^{UI}(2)$$

so, from the user's view-point, effort should be made to keep the component 2 at state 4. i.e the state 4 of component 2 has the highest contribution to the system.

Case 2:

By using the theorem, the utility importance of component 1 is as follows :

∗ At level : $m = M = 4$

$$I_4^{UI}(1) = p_{1,4}\left\{\sum_{j=0}^{3} a_j \Delta_{4,4,4}^j + a_4 L_{4,4}^4\right\}$$

$$\Delta_{4,4,4}^1 = H_3^1(4_1,4) + H_4^1(4_1,4)$$

$$H_3^1(4_1,4) = [(4110)] + [(4210,4120)] + [(4310,4130)] + [(4410,4140)]$$

$$H_4^1(4_1,4) = [(4111)] + [(4211,4121,4112)] + [(4311,4131,4113)] + [(4411,4141,4114)]$$

$$\Delta_{4,4,4}^2 = H_3^2(4_1,4) + H_4^2(4_1,4)$$

$$H_3^2(4_1,4) = [(422\times)] + [(432\times,423\times)] + [(442\times,424\times)] \quad \times \text{can be } 0,1$$

$$H_4^2(4_1,4) = [(4222)] + [(4322,4232,4223)] + [(4422,4242,4224)]$$

$$\Delta_{4,4,4}^3 = H_3^3(4_1,4) + H_4^3(4_1,4)$$

$$H_3^3(4_1,4) = [(433\times)] + [(443\times,434\times)] \quad \times \text{can be } 0,1,2$$

$$H_4^3(4_1,4) = [(4333)] + [(4433,4343,4334)]$$

$$L_{4,4}^4 = [(444\times)] + [(4444)] \quad \times \text{can be } 0,1,2,3$$

so:

$$I_4^{UI}(1) = 37.104$$

∗ At level : $m = 0$

$$I_0^{UI}(1) = p_{1,0}\left\{a_0[L_{4,0}^0 + \Delta_{4,0,1}^*] + \sum_{j=1}^{4} a_j [L_{3,0}^j + \Delta_{3,0,j+1}^j]\right\}$$

$$L_{3,0}^1 = [(0111)] \quad , \quad L_{3,0}^2 = [(0222)] \quad ,$$

$$L_{3,0}^3 = [(0333)] \quad , \quad L_{3,0}^4 = [(0444)]$$

$$\Delta_{3,0,2}^1 = H_3^1(0_1,2) + H_3^1(0_1,3) + H_3^1(0_1,4)$$

$$H_3^1(0_1,2) = [(0211,0121,0112)] + [(0221,0212,0122)]$$

$$H_3^1(0_1,3) = [(0311,0131,0113)] + [(0321,0312,0132,0123,0213,0231)] + [(0331,0313,0133)]$$

$$H_3^1(0_1,4) = [(0411,0141,0114)] + [(0421,0241,0142,0412,0124,0214)] + [(0431,0341,0143,0413,0134,0314)] + [(0441,0414,0144)]$$

$$\Delta_{3,0,3}^2 = H_3^2(0_1,3) + H_3^2(0_1,4)$$

$$H_3^2(0_1,3) = [(0322,0232,0223)] + [(0332,0323,0233)]$$

$$H_3^2(0_1,4) = [(0422,0242,0224)] + [(0432,0423,0342,0324,0234,0243)] + [(0442,0424,0244)]$$

$$\Delta_{3,0,3}^3 = H_3^3(0_1,4) = [(0433,0343,0334)] + [(0443,0434,0344)]$$

then:

$$I_0^{UI}(1) = 6.551$$

∗ At level: $m = 3 < M = 4$

$$I_3^{UI}(1) = p_{1,3}\left\{\begin{array}{l} a_0\Delta_{4,3,3}^0 + a_1\Delta_{4,3,3}^1 + a_2\Delta_{4,3,3}^2 + \\ a_3[L_{4,3}^3 + \Delta_{4,3,4}] + a_4 L_{3,3}^4 \end{array}\right\}$$

$$\Delta_{4,3,3}^1 = [H_3^1(3_1,3) + H_3^1(3_1,4)] + [H_4^1(3_1,3) + H_4^1(3_1,4)]$$

$$H_3^1(3_1,3) = [(3110)] + [(3210,3120)] + [(3310,3130)]$$

$$H_3^1(3_1,4) = [(3410,3140)]$$

$$H_4^1(3_1,3) = [(3111)] + [(3211,3121,3112)] + [(3311,3131,3113)]$$

$$H_4^1(3_1,4) = [(3411,3141,3114)]$$

$$\Delta_{4,3,3}^2 = [H_3^2(3_1,3) + H_3^2(3_1,4)] + [H_4^2(3_1,3) + H_4^2(3_1,4)]$$

$$H_3^2(3_1,3) = [(322\times)] + [(332\times,323\times)] \times \text{can be } 0,1$$

$$H_3^2(3_1,4) = [(342\times,324\times)] \quad \times \text{can be } 0,1$$

$$H_4^2(3_1,3) = [(3222)] + [(3322,3232,3223)]$$

$$H_4^2(3_1,4) = [(3422,3242,3224)]$$

$$L_{4,3}^{3}=[(333\times)]+[(3333)] \quad \times \text{can be } 0,1,2$$

$$\Delta_{4,3,4}^{*}=H_{3}^{3}(3_1,4)+H_{4}^{3}(3_1,4)$$

$$H_{3}^{3}(3_1,4)=[(334\times,343\times)]+[(344\times)]$$

$$\times \text{can be } 0,1,2$$

$$H_{4}^{3}(3_1,4)=[(3334,3343,3433)]+ [(3344,3434,3443)]$$

$$L_{3,3}^{4}=R_4(3_1,3,4)=[(3444)]$$

so:

$$I_{3}^{UI}(1)=328.576$$

we find:

$$I_{3}^{UI}(1)>I_{4}^{UI}(1)>I_{0}^{UI}(1)$$

so, from the user's view-point, effort should be made to keep the component 1 at state 3. i.e the state 3 of component 1 has the highest contribution to the system.
In this example, with the data above such that:

$p_{2,3}>p_{2,4}>p_{2,0}$, $\qquad p_{1,3}>p_{1,4}>p_{1,0}$

and

$$(a_0,a_1,a_2,a_3,a_4)=(0,100,1000,2000,8000)$$

we find that :

$$I_{4}^{UI}(2)>I_{3}^{UI}(2)>I_{0}^{UI}(2) \quad \text{and} \quad I_{3}^{UI}(1)>I_{4}^{UI}(1)>I_{0}^{UI}(1)$$

However, the component 2 and the component 1 have the greatest probability in state 3 $(p_{2,3}=0.3,\ p_{1,3}=0.4)$, but it hasn't a big influence on their importance utility. So, one can see that the position of a component in a M.S consecutive k -out-of- $n:G$ system is important and has the most effect on the component importance utility.

CONCLUSION

In this paper, we present a method for computing the performance utility-function of a M.S consecutive k -out-of- $n:G$ system. Since this function depends directly on the importance utility of the components $(I_m^{UI}(i), i=1,...,n)$, so the theorem and the corollary give $I_m^{UI}(i)$ of component i, and we have seen that the position of component i is very important and has a great effect on the results as shown in the example. Of course, the values of $p_{i,j}$ and $a_j\,(i=1,...,n,\quad j=0,1,...,M)$ are taken into consideration. In other words, by ignoring the performance utility levels and the probability distribution of the components, it is impossible to define a meaningful index to measure the performance utility of an individual component.

REFERENCES

[1] J. Fussell, How to calculate system reliability and safety characteristics, IEEE Trans.Reliab.,vol. 24, n° 3 (1975), pp. 169-174.

[2] M. Cheok, G. Parry, and R. Sherry, "Use of importance measures in risk informed applications," Reliab. Eng. Syst. Saf., vol. 60, (1998), pp. 213-226 .

[3] A. Aggarwal and R. Barlow, "A survey on network reliability and domination theory," Oper. Res., vol. 32, n° 2 (1984), pp. 478-492.

[4] E. Elsayed, Reliability Engineering: Addison Wesley Longman Inc., (1996).

[5] A. Hoyland and M. Rausand, System Reliability Theory: Models and Statistical Methods: John Wiley & Sons (1994).

[6] F.C. Meng, "Comparing the importance of system components by some structural characteristics," IEEE Trans. Reliab., vol. 45, n° 1 (1996), pp. 59-65.

[7] F.C. Meng, "Some further results on ranking the importance of system components," Reliab. Eng. Syst. Saf., vol. 47 (1995), pp. 97-101.

[8] J.S. Hong and C.H. Lie, "Joint reliability-importance of two edges in an undirected network," IEEE Trans. Reliab., vol. 42, n° 1 (1993), pp. 17-23.

[9] M.J. Armstrong, "Joint reliability-importance of components," IEEE Trans. Reliab., vol. 44, n° 3 (1995), pp. 408-412.
[10] A.P. Wood, "Multistate block diagrams and fault trees," IEEE Trans. Reliab., vol. R-34, n° 3 (1985), pp. 236-240.
[11] G. Levitin and A. Lisnianski, Multi-state System Reliability. Series on Quality, Reliability and Engineering Statistics-vol. 6 : World Scientific Publishing (2003).
[12] E. El-Neweihi, F. Proschan, and J. Sethuraman, "Multi-state coherent systems," J. Appl. Prob., vol. 15 (1978), pp. 675-688 .
[13] W. Barlow and A. Wu, "Coherent systems with multi-state components," Math.Oper. Res., vol. 3, n° 4 (1978), pp 275-281.
[14] W. Griffith, "Multi-state Reliability models," J. Appl. Prob., vol. 17 (1980), pp. 735-744 .
[15] G. Levitin and A. Lisnianski, "Importance and sensitivity analysis of multi-state systems using the universal generating function," Reliab. Eng. Syst. Saf., vol. 65 (1999) pp. 271-282 .
[16] E. Zio and L. Podofillini, "Monte-Carlo simulation analysis of the effects on different system performance levels on the importance on multi-state components," Reliab. Eng. Syst. Saf., vol. 82 (2003), pp. 63-73.
[17] J.E. Ramirez-Marquez and D.W. Coit,"Composite Importance Measures for Multi-state Systems with Multi-state Components," IEEE Trans. Reliab., vol.54, n° 3 (2005), pp.517--529, .
[18] S. Wu and L.Y. Chan, "Performance Utility-Analysis of Multi-State Systems,"IEEE. Trans.Reliab.,vol.52, n° 1 (2003), pp.14-21.
[19] S. Belaloui and B. Ksir,"Reliability of multi-state consecutive k -out-of- $n : G$ systems," Inter .Journ.of. Reliab., Quality and Safety Engineering, vol.14, n° 4 (2007), pp.361-377.
[20] J. Huang, M.J. Zuo and Z. Fang,"Multi-state consecutive k -out-of- n systems," IIE Trans. on Reliab. Engineering. vol.10, n° 3 (2003), pp. 345-358 .