Sciences & Technologie A – N°21, juin (2004), pp. 11-14.

ON THE LOCAL TIME OF A SEMI-STABLE PROCESS

Received 22/06/2003 – Accepted 15/05/2004

Abstract

In this paper we develop the note [2]. However, we use a different method based on a formula which was established by one of us in [3].

***Keywords**: Local time, entropy, stable subordinator.*

Résumé

Dans cet article, est développée la référence [2]. Cependant, une méthode différente est utilisée basée sur l'expression établie dans la référence [3].

***Mots clés**: Temps local, entropie, subordinateur stable.*

1991 M.S.C.: Primary 60J55, Secondary 60G17.

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ملخص

في هذا العمل نطور البحث [2]. لكن هنا نستعمل برهانا جديدا، الذي يرتكز على علاقة أثبتها أحدنا في البحث [3].

1- INTRODUCTION

Let $X(t)$ be a semi-stable process with indices (α,υ), $\alpha \in]1,2]$ and $\upsilon \in]-1,\infty[$, see [11]. For x_0 in R let $Z(x_0,t) = \{s \le t \,/\, X(s) = x_0\}$ and $L(t)$ its (Blumenthal-Getoor) local time. Our aim is to derive the entropy-like limit theorem of $L(t)$ given in [2] in a succinct way, which turns out to be instructive and in the same time provides a detailed proof of the note [2]. Indeed, a complete proof appears only in the Ph. D. thesis of one of us. Our argument here is based on a formula which was established in [3] by a lengthy discretization procedure, see theorem 1 below. For the sake of completeness, the formula itself is given a shorter proof here.

Let $I_{n,k}$, $k > 0$, be the interval $[k2^{-n}, (k+1)2^{-n}[$. For any bounded set of real numbers E, let $N(n, E)$ be the number of these intervals that intersect E. The number $N(n,Z(x_0,t))$, or simply $N(n, t)$, obviously increases to infinity as $n \uparrow \infty$ a.s. and we shall study the non-trivial limit behaviour of the entropy functional

$$K(n,t) = N(n, t) \,/\, f(n) \qquad (1.1)$$

for a suitable normalizing factor f(n) a.s. It turns out that the limit is local time.

1.1- Relevance to Monte-Carlo methods and building science

In comparison with the following existing constructions : Bally [1], Getoor [5], Getoor and Millar [6], Griego [7]. Kingman [9], Taylor and Fristedt [12], Taylor and Wendel [13], Williams [14], our method, besides being intrinsic, is particularly well suited to numerical computations by Monte-Carlo methods, because there is no need to dispose *of all* the level times s s.t. $X(s) = x_0$ in order to compute the functional $K(n, t)$, once the grid $I_{n,k}$, $k \ge 0$, is installed. It is therefore possible, in principle, to use our construction to simulate resolvents. This numerical side of our construction should prove useful to the general reader of the present journal, alongside other theoretical sides. Moreover, see remark one, material science engineers, for example, and statisticians should find theorem one a source of new study.

The literature that is necessary for the main ideas is available in the library of the department of mathematics of Constantine, for example. Unimportant constants are designated by c, c'…and these may vary from line to line while proofs are in process.

2- PRELIMINARIES

2.1- Subordinators

The right continuous inverse of $L(t)$, denoted $S(t)$, is given by $S(t)=\inf\{s\geq 0/L(s)>t\}$. The strong Markov property of X (in the sequel we suppose X starts from x_0 a.s.) shows that $S(t)$ is a subordinator. That is, a process with stationary independent increments defined on some probability space (Ω,F,P).The sample paths are a.s. right continuous with left-hand limits. The Laplace transform of S is given by

$$E\exp-\lambda\delta^{\frac{1}{\beta}}\left[S(t_2)-S(t_1)\right]=\exp-(t_2-t_1)g(\lambda), \qquad (2.1)$$

where the *exponent* $g(\lambda)={}^{\lambda\beta}$ and the *index* $\beta=\dfrac{\alpha-1}{\alpha+\upsilon}$ lies in]0, 1[, $\delta>0$, $\lambda\geq 0$, $t_2\geq t_1\geq 0$, with $S(0)=0$ a.s. The Lévy measure is absolutely continuous and is given by $\gamma(dx)=cx^{-1-\beta}dx$, where $c=\left[\delta\Gamma(2-\beta)\right]^{-1}$, see Williams [15]. For $x>0$, the finite number $\gamma((x,\infty))$ is designated by $H(x)$. A related quantity of interest in this paper is the *mean value* of $H(x)$ over small intervals

$$\overline{H(x)}=\frac{1}{x}\int_0^x H(y)dy=cx^{-\beta}.$$

It indeed is possible to invert the transform (2.1), see [10]. So that the laws $P_t(dx)$ of the random variables $S(t)$, which form a weakly continuous convolution semi-group and then induces a Feller transition semi-group, are absolutely continuous with densities $f(t, x)$. The scaling property of S shows that $f(t,x)=t^{\frac{-1}{\beta}}f(1,xt^{\frac{-1}{\beta}})$, $t>0$, $x>0$.

The stopped potential operator $G_t(dx)$ is also absolutely continuous with density $q_t(x)=\int_0^t f(\tau,x)d\tau$. It is clear that $q_t(x)$ is monotone decreasing in x for all $t>0$ and that $G_t(R_+)=t$. In the sequel denote, for $x\in I_{n,0}$, put $x(i)=x+i2^{-n}$ and $G_t(x)=\int_0^x q_t(y)dy$. The following Lemma is straightforward.

Lemma 1. *With the notations above, we have for* $x\in I_{n,0}$

$$\sum_{i\geq 0}q_t(x(i))=2^n\left[t-G_t(x)+G_t^*(x)\right]$$

where $G_t(x)<c2^{-n}$ and $G_t^*(x)\leq 2^{-n}q_t(x)$.

2.2- A key formula

In addition to the proof in [3] to derive the formula below, we note that it is also a consequence of the Lévy-system master-formula for $S(t)$.

Theorem 1. *Let* $b>0$, $c\in\left]b,\infty\right]$ *and* $T>0$, *then*

$$P(\exists t\leq T/S(t)\in\left[b,c\right[)=\int_0^b\gamma(\left[b,c\right[-x)G^\tau(dx).$$

Proof

Following theorem 35 in p. 253 of Dellacherie and Meyer [4] and letting J stand for the set of jumps of our process, we have the identity

$$E\left[\sum g_t f(S_{t-},S_t)\right]=E\left[\int_0^\infty g_t\overline{N}f(S_{t-})dt\right],$$

where g_t is a positive previsible process, $f(.,.)$ a Borel measurable function on $R_+\times R_+$ and $\overline{N}f(x)$ the kernel arising from the tensor product of measures $\overline{N}f(x)=\delta_x\times N(x,dy)$, in which, for positive measurable functions φ, the kernel $N(x,dy)$ is given by $N_{\varphi(x)}=\int_0^\infty\varphi(x+y)\gamma(dy)$. It suffices to take, since the jump that leads $S(t)$ into the interval $[b, c[$ before T occurs only once in $[0, T]$, the deterministic indicator $g_t=I_{]0,t]}$ and $f(x,y)=I_A$, where A = $]0,b[\times[b, c[$, and to bear in mind that any fixed time $t\geq 0$ is a continuity point of S, so that $P_{S_{t-}}(dx)$ is just $P_{S_t}(dx)$ to yield the result. □

3- THE ENTROPY CONSTRUCTION OF *L*(*t*)

Let us put $f=\overline{H}$ in section (1). Our main result is the following theorem in which, by abuse of notation allowed by a well known time-substitution procedure, the functional $K(n, T)$ rather refers to the range $S([0, T])$ of our subordinator,

Theorem 2. *We have* $\forall T>0$, $\lim_{n\to\infty}K(n,T)=cT$ *a.s.* ,

where $c=\left[\delta\Gamma(2-\beta)\right]^{-1}$

Fix $T>0$. We first have,

Lemma 2. *We have for fixed* $T>0$,

$$EN(n,T)=c2^{n\beta}T+Q_{n,T}^1,\ \textit{where}\left|Q_{n,T}^1\right|\leq c'.$$

Proof

We have by Theorem 1

$$EN(n,T)=1+\sum_{i\geq 1}P(\exists t\leq T/S(t)\in I_{n,i})$$
$$=1+\sum_{1\leq i}\int_{\left[0,i2^{-n}\right[}\gamma(I_{n,i}-x)G_T(dx)$$
$$=1+\sum_{i\geq 0}\int_{I_{n,i}}H\left[(i+1)2^{-n}-x\right]G_T(dx)$$
$$=1+\int_{I_{n,0}}H(2^{-n}-x)\left[\sum_{i\geq 0}q_T(x(i))\right]dx$$

and the Lemma follows. □

Let us consider now the second moment

Lemma 3. *We have for fixed* $T>0$:

$$EN^2(n,T)=(c2^{n\beta}T)^2+O(2^{n\beta}).$$

Proof

We clearly have

$$EN^2(n,T)=2\sum_{1\le i<j}P(\exists t,t'\le T/S(t)\in I_{n,i}\quad and\quad S(t')\in I_{n,j})$$
$$=1+3\sum_{1\le k}P(\exists t\le T/S(t)\in I_{n,k.}).$$

Let us look at the first sum on the right hand side above. We apply the strong Markov property (in a special form, see e.g. [4] p. 179), we have

$$P(\exists t,t'\le T/S(t)\in I_{n,i}\quad and\quad S(t')\in I_{n,j})$$
$$=\int_{[0,T]\times I_{n,i}}F_i(d\tau,dz)\int_0^{j2^{-n}-z}\gamma(I_{n,j}-z-x)G_{T-\tau}(dx)$$

where $F_i(d\tau,dz)$ is the joint law of the couple $(\tau_{I_{n,i}},S(\tau_{I_{n,i}}))$ in which $\tau_{In,i}$ is the hitting time of $I_{n,i}$. Hence

$$\sum_{1\le i<j}P(\exists t,t'\le T/S(t)\in I_{n,i}\quad and\quad S(t')\in I_{n,j})$$
$$=\sum_{1\le i}\int_{[0,T]\times I_{n,0}}F_i(d\tau,i2^{-n}+dz')\sum_{1\le j}\int_0^{j2^{-n}-z'}\gamma(I_{n,j}-z'-x)G_{T-\tau}(dx)$$
$$=\sum_{1\le i}\int_{[0,T]\times I_{n,0}}F_i(d\tau,i2^{-n}+dz')\sum_{1\le j}\int_{I_{n,0}}H(2^{-n}-x')G_{T-\tau}(j2^{-n}-z'+dx')$$
$$+\sum_{1\le i}\int_{[0,T]\times I_{n,0}}F_i(d\tau,i2^{-n}+dz')\int_0^{2^{-n}-z'}H(2^{-n}-z'-x)G_{T-\tau}(dx).\tag{3,1}$$
$$=\sum_{1\le i}\int_{[0,T]\times I_{n,0}}F_i(d\tau,i2^{-n}+dz')\sum_{1\le j}\int_{I_{n,0}}H(2^{-n}-x')$$
$$\times q_{T-\tau}(j2^{-n}-z'+x')dx'+Q^2_{n,T},$$

where obviously $\left|Q^2_{n,T}\right|\le c2^{n\beta}$. Now the main term in the right hand side of (3.1) becomes

$$\sum_{1\le i}\int_0^T(T-\tau)d\left[\int_0^{i2^{-n}}\gamma(I_{n,i}-x)G_\tau(dx)\right]\times\overline{H}(2^{-n})+Q^3_{n,T},\tag{3,2}$$

where $\left|Q^3_{n,T}\right|\le c2^{n\beta}$.

An integration by parts in the main term in (3.2) gives $\frac{1}{2}\left[\overline{H}(2^{-n})T\right]^2+Q^4_{n,T}$, where $\left|Q^4_{n,T}\right|\le c2^{n\beta}$. Hence

$$2\sum_{1\le i<j}P(\exists t,t'\le T/S(t)\in I_{n,i}\text{ and }S(t')\in I_{n,j})=\left[\overline{H}(2^{-n})T\right]^2+Q^5_{n,T},$$

where $Q^5_{n,T}=Q^2_{n,T}+Q^3_{n,T}+Q^4_{n,T}$ and as $n\to\infty$, $\left|Q^5_{n,T}\right|=O(2^{n\beta})$ from which the Lemma follows. □

We are now in the position to prove Theorem 2.

Proof

For fixed T > 0, it suffices to use a standard Borel-Cantelli argument. Then taking advantage of the monotonicity of $N(n, T)$ with respect to time, we can easily derive an almost sure statement for all T> 0. □

Corollary 1. *We have a.s.*

$$\forall t>0,\ \lim_{n\to\infty}K(n,T)=\left[\delta\Gamma(2-\beta)\right]^{-1}L(t).$$

Proof

Recall that K (n, t) was defined in Section (1). We need only put $T = L(t)$ in Theorem 2. □

4. RELATION TO OTHER LIMIT THEOREMS

In fact $N(n, t)$ is the result of two actions. On the one hand, the excursions of duration larger than 2^{-n}, accomplished before t, contribute $N_0(n,t)$ intervals $I_{n,k}$. On the other hand, the build up of the smaller excursions also reaches new intervals $I_{n,k}$. Let us compute the contribution of the small excursions. Using a well known limit theorem due to Lévy we deduce from the equality

$$\frac{N(n,t)}{\overline{H}(2^{-n})}=\frac{N_0(n,t)}{H(2^{-n})}\left[\frac{H(2^{-n})}{\overline{H}(2^{-n})}\right]+\frac{R(n,t)}{\overline{H}(2^{-n})}.$$

where R(n, t) is the number of intervals $I_{n,k}$, $k\ge 0$, entered by the subordinator before *t > 0* but *are not* those which are hit by a big jump $\ge 2^{-n}$, that $\frac{R(n,t)}{\overline{H}(2^{-n})}$ contributes a portion of local time, a.s. . In the special case of Brownian motion, i.e. α = 2 and ν = 0. this implies that the big excursions have *exactly* the same contribution as the smaller ones. This underlines, once more, the symmetries in Brownian motion. Actually, these small excursions contribute to local time through an alternative entropy quantity, namely $r(n,t)=\sum_{I<2^{-n}}I$, where I designates the length of an excursion registered before t > 0. Under suitable conditions, see Ito and Mc Kean [8] p. 219 or Taylor and Fristedt [12], certainly satisfied here, the right corrector is here the number

$$\int_0^{2^{-n}}x\gamma(dx)=2^{-n}\left[\overline{H}(2^{-n})-H(2^{-n})\right].$$

Remark 1

We believe that subordinators may well model the evolution of cracklings inside building materials e.g.. This is especially reliable if the fissure is the result of the action of independent agents or perhaps a single agent which is renewing independently in time and the material is somewhat brittle. Theorem 1 should prove valuable as it does supply a workable expression for the distribution, in time, of critical thresholds.

Remark 2

Although the construction of Taylor and Fristedt [12] *is of profound scope, it does not give our limit theorem* *because the functional K(n, t), due to its very nature of controlling* $Z(x_0,t)$*, is not an obvious by-product of excursion statistics. The Kingman limit law being universal, is of considerable theoretical importance but still is not quite adequate for numerical computations and nor does it give our limit procedure.*

Remark 3

The proof above extends to regularly varying exponents with index $0<\beta<1$*. i.e.* $g(\lambda)=\lambda^{\beta}l(\lambda)$ *(and even to quite general Markov processes provided the resolvent is absolutely continuous with density q(x) that is a.e. continuous). In particular, if the slowly varying function l(x) is quasi-monotone, then a.s. for all t >0*

$$\lim_{n\to\infty}\frac{N(n,t)}{\left(\dfrac{2^{n\beta}l(2^n)}{\Gamma(2-\beta)}\right)}=L(t).$$

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