MIXED FINITE ELEMENT FOR CRACKED INTERFACE

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Résumé

Un élément spécial basé sur le principe variationnel mixte de Reissner est présenté pour étudier la rupture interfacielle dans les bimatériaux. C'est un élément fini mixte bidimensionnel à 7 nœuds avec 5 nœuds déplacement et 2 nœuds contrainte. Cet élément assure la continuité des vecteurs déplacement et contrainte sur la partie cohérente et la discontinuité de celle-ci sur la partie fissurée. Cet élément d'interface est associé à la méthode d'extension virtuelle de fissure pour calculer le taux de restitution d'énergie. Les résultats obtenus, avec l'élément d'interface présenté, montrent une bonne concordance avec les solutions.

Mots clés *: Elément fini mixte d'interface, Interface fissurée, Taux de restitution d'énergie, Méthode d'extension virtuelle de fissure*

Abstract

A special finite element based on Reissner's mixed variational principle has been presented to study interfacial cracks in bimaterials. The present element is a 7-node two dimensional mixed finite element with 5 displacement nodes and 2 stress nodes. The mixed interface finite element ensures the continuity of stress and displacement vectors at the interface on the coherent part and the discontinuity of this one on the cracked part. This interface element was associated with the virtual crack extension method to evaluate the energy release rates using only one meshing by finite elements. Results obtained from the present mixed interface element have been shown to be in good agreement with the analytical solutions.

Keys words *: Mixed interface finite element; Cracked interface; Energy release rate; Virtual crack extension*

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ملخص

في هذه الدراسة قمنا بتقديم عنصر متناهي انطلاقا من مبدأ التغيرات المختلطة لريزنر. هذا العنصر المتناهي ذو بعدين مؤلف من 7 عقد منها 5 عقد للانسحابات و عقدتين للإجهادات. العنصر المقترح يضمن استمرار أشعة الانسحابات و الإجهادات في الجزء الملتصق و عدم الاستمرار في الجزء المتشقق. هذا العنصر الفاصل تم استعماله مع طريقة التمدد الافتراضي للانشقاق من أجل حساب معامل استعادة الطاقة. النتائج المتحصل عليها, باستعمال العنصر المقترح, تبين التقارب الجيد لها مع الحلول النظرية

الكلمات المفتاحية: *العنصر المتناهي المختلط للفاصل, الفاصل المتشقق, معامل استعادة الطاقة, طريقة التمدد الافتراضي للانشقاق.*

Several disorders observed in an existing work on civil engineering may have their origin in local phenomena which reveal the weak points of this work. These critical zones are located, on the one hand in the links between materials or interfaces, on the other hand in singularly formed areas such as cavities, angles and cracks, seats of strong stress concentrations.

In this paper, the mixed finite element method is used for the study interfacial cracks in bimaterials. The mixed variational formulation has several advantages [1-2] over the conventional finite element formulations (specifically the displacement method), including direct evaluation of nodal stresses along with nodal displacements; improved accuracy of both displacements and stresses, and adequacy of lower-order elements, leading to elegant grids of discretization.

The mixed finite element method developed by Herrmann [3] for plate bending analysis has been extended to plane elasticity problems by Mirza and Olson [4]. An exhaustive literature on mixed finite element models has been compiled by Noor [5]. Aivazzadeh [6] developed a family of rectangular mixed interface element using Reissner's mixed variational principle. Habib [7] presented various axisymmetric mixed element for studying bonded assemblies and laminate structure. Bichara [8] and Sarhane-Bajbouj [9] developed mixed finite elements for one or multi interfaces. Wu and Lin [10] presented a two dimensional mixed finite element scheme based on a local high-order displacement model for the analysis of sandwich structure. Carrera [11-13] also presented various mixed models based on Reissner's mixed variational principle. Ramtekkar and al. [14] developed a three dimensional mixed finite element model using the minimum potential energy principle. This model has been used for the analysis of sandwich plates [15]. Desai and Ramtekkar [16] presented a mixed finite element based on displacement theory satisfying fundamental elasticity relations. Bambole and Desai [17] developed a two-dimensional hybrid-interface element based on the principle of minimum potential energy.

In this work a mixed finite element model has been presented using Reissner's mixed variational principle. The model takes into account the continuity of the interface on the coherent part (mechanical and geometrical continuity) and the discontinuity of this one on the cracked part (edge effect). This mixed finite element was developed by Bouzerd [18] using a direct formulation: the shape functions of the displacement and stress fields are built directly starting from the real configuration of the element in a physical (x, y) plane.

In the present paper, this element was reformulated starting from a parent element in a natural (ξ,η) plane. This formulation presents, in addition to the simplification of calculations, the enormous advantage of modelling the different types of cracks with various orientations. This interface element was associated with the virtual crack extension method to evaluate the energy release rates using only one meshing by finite elements.

The accuracy of the element has been evaluated by comparing the numerical solution with an available analytical solution or numerical ones obtained from others finite elements.

1. FORMULATION OF THE INTERFACE ELEMENT

The stages of construction of the proposed interface element are schematized on figure 1.

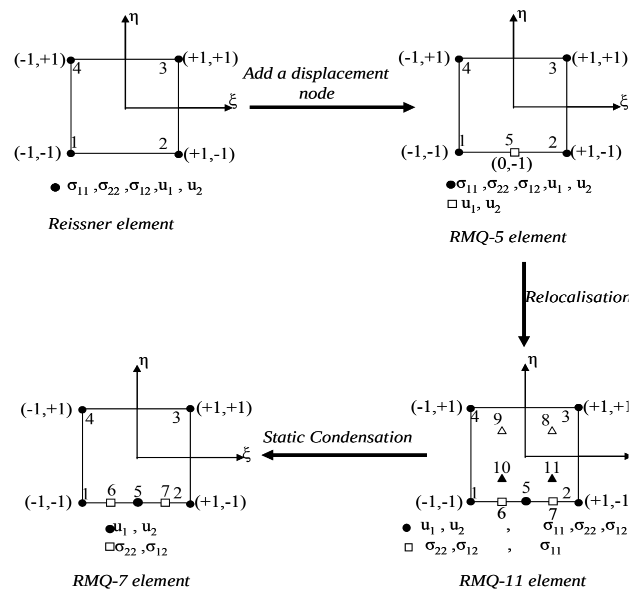

Figure 1 : Stages of construction of RMQ-7 element

The RMQ-7 (Reissner Modified Quadrilateral) element is a quadrilateral mixed element with 7 nodes and 14 degrees of freedom [18]. Three of its sides are compatible with linear traditional elements and present a displacement node at each corner. The fourth side, in addition to its two displacement nodes of corner (node 1 and node 2), offers three additional nodes: a median node (node 5) and two intermediate nodes in the medium on each half-side (nodes 6 and 7), introducing the components of the stress vector along the interface.

The Continuity of the displacement and stress vectors can be taken into account on the level on this particular side, which must be placed along the interface. In the cracked structures, the median node is associated to the point of crack; the two static nodes on both sides make it possible to meet the two essential requirements of such a situation, which are the free edge condition on the lips of the crack and the conditions of continuity along the coherent part.

At the beginning, we start with Reissner's mixed formulation with all displacements and all stresses like nodal variables to build the interface mixed element. There are thus surplus nodal variables. This formulation imposes a too strong continuity, indeed the stress σ_{11} figure among the variables considered in the Reissner variational functional, but does not appear among the interface stresses (separation stress σ_{22} and shear stress σ_{12}); therefore we will eliminate this stress (σ_{11}) in the formulation of the interface element.

1.1. Construction of the parent element RMQ-5

The RMQ-5 element is obtained by adding a displacement node to the Reissner mixed element. It is a mixed element with 5 nodes and 22 degrees of freedom. It has a side (associate with the interface) presenting three nodes, the medium node (displacement node) characterizes the bottom of the crack in the final version of the element. The stress nodes not having changed neither number some, nor in position, do the RMQ-5 and the Reissner elements present the same static behaviour. The stress field is expressed by the same shape functions.

The element displacement component is approximated by :

$$\{u\} = [N]\,\{q\} \tag{1}$$

where : $\{q\}^t = \{u_1^1, u_2^1, u_1^2, u_2^2, u_1^3, u_2^3, u_1^4, u_2^4, u_1^5, u_2^5\}$ is the vector of nodal displacements and $[N]$ is the matrix of interpolation functions for displacements.

$$[N] = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 & 0 & N_5 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 & 0 & N_5 \end{bmatrix} \tag{2}$$

The shape functions are :

$$N_1 = -\frac{1}{4}(1-\xi)(1-\eta)\xi \;,\quad N_2 = \frac{1}{4}(1+\xi)(1-\eta)\xi \;,\quad N_3 = \frac{1}{4}(1+\xi)(1+\eta)$$

$$N_4 = \frac{1}{4}(1-\xi)(1+\eta) \;,\quad N_5 = \frac{1}{2}\left(1-\xi^2\right)(1-\eta) \tag{3}$$

The stress field in any point is written :

$$\{\sigma\} = [M]\,\{\tau\} \tag{4}$$

where $[M]$ is the matrix of interpolation functions for stresses, and

$\{\tau\}^t = \{\sigma_{11}^1, \sigma_{22}^1, \sigma_{12}^1, \sigma_{11}^2, \sigma_{22}^2, \sigma_{12}^2, \sigma_{11}^3, \sigma_{22}^3, \sigma_{12}^3, \sigma_{11}^4, \sigma_{22}^4, \sigma_{12}^4\}$ vector of nodal stresses

The nodal approximation of the displacement and stress fields is expressed by :

$$\begin{Bmatrix} \{\sigma\} \\ \{\varepsilon\} \end{Bmatrix} = \begin{bmatrix} [M] & [0] \\ [0] & [B] \end{bmatrix} \begin{Bmatrix} \{\tau\} \\ \{q\} \end{Bmatrix} \tag{5}$$

where $[B]$ is the strain-displacement transformation matrix.

The element matrix $[K_e]$ is given by :

$$[K_e] = \begin{bmatrix} [K_{\sigma\sigma}] & [K_{\sigma u}] \\ [K_{\sigma u}]^t & [0] \end{bmatrix} \tag{6}$$

Here : $$[K_{\sigma\sigma}] = -\,e \int_{A^e} [M]^t [S]\,[M]\,dA^e \tag{7}$$

and $$[K_{\sigma u}] = e \int_{A^e} [M]^t [B]\,dA^e \tag{8}$$

where: e is the thickness, $[S]$ is the compliance matrix and A^e is the element area.

1.2. Construction of the RMQ-11 element

The RMQ-11 element is obtained starting from the parent element RMQ-5 by relocalisation [19] of certain variables inside the element and by displacement of static nodal unknown of the corners towards the side itself. It is an element with 11 nodes and 22 degrees of freedom. The displacements nodes not having changed neither number some, nor in position, do the elements RMQ-5 and RMQ-11 present the same shape functions.

The approximation of the stress field according to the nodal variables $\{\tau\}$ is :

$$\sigma(\xi,\eta) = \{P(\xi,\eta)\}[P_n]^{-1}\{\tau\} = [M]\{\tau\} \tag{9}$$

where polynomial base of the element is :

$$\{P(\xi,\eta)\} = \{1 \quad \xi \quad \eta \quad \xi\eta\}$$

and $[P_n]$ is the nodal matrix.

In the configuration of figure 1, the shape functions, used to approximate σ_{11}, are given by :

$$M_{11}^8 = \frac{1}{4}(1+2\xi)(1+2\eta) \;,\qquad M_{11}^9 = \frac{1}{4}(1-2\xi)(1+2\eta)$$

$$M_{11}^{10} = \frac{1}{4}(1-2\xi)(1-2\eta) \;,\qquad M_{11}^{11} = \frac{1}{4}(1+2\xi)(1-2\eta) \tag{10}$$

The shape functions used to evaluate σ_{22} and σ_{12} are given as follows**:**

$$M_{i2}^6 = \frac{1}{6}(1-2\xi)(1-2\eta) \;,\qquad M_{i2}^7 = \frac{1}{6}(1+2\xi)(1-2\eta)$$

$$M_{i2}^8 = \frac{1}{3}(1+2\xi)(1+\eta) \;,\qquad M_{i2}^9 = \frac{1}{3}(1-2\xi)(1+\eta) \tag{11}$$

The element stiffness matrix is written in the form given by the expressions (6), (7) and (8). This matrix can be evaluated by Gauss numerical integration scheme, with four points (2x2) on the element.

1.3 Construction of the RMQ-7 element

The four intern's nodes of RMQ-11 element who do not take part in the assembly contain degrees of freedom and inappropriate surplus variables with the interface and the free edge to supplement the set of variables in the polynomial interpolation of the stress field; what has as a consequence of many nodes and variable in kind. In practice these nodes complicate the operation of setting in data, and increase the size half-width of band during the assembly, which causes an increase of the computing time. The method used for condensation of the internal degrees of freedom to contour is related to the general concept of reduction of the size of an equations system per elimination of a certain number of variables. Gallagher [20] used this type of procedure in structural analysis. The static condensation procedure leads to the following reduced elementary matrix :

$$[K_e] = \begin{bmatrix} [K_{\sigma\sigma}]^* & [K_{\sigma u}]^* \\ [K_{u\sigma}]^* & [K_{uu}]^* \end{bmatrix} \tag{12}$$

where $$[K_{\sigma\sigma}]^* = [K_{\sigma\sigma}]_c - [K_{\sigma\sigma}]_{ci}[K_{\sigma\sigma}]_i^{-1}[K_{\sigma\sigma}]_{ci}^t \tag{13}$$

$$[K_{\sigma u}]^* = [K_{\sigma u}]_c - [K_{\sigma u}]_i [K_{\sigma u}]_i^{-1} [K_{\sigma u}]_i^t \tag{14}$$

$$\left[K_{u\sigma}\right]^* = \left[K_{\sigma u}\right]_i^{*t} \quad (15)$$

and $$\left[K_{uu}\right]^* = -\left[K_{\sigma u}\right]_i \left[K_{\sigma\sigma}\right]_i^{-1} \left[K_{\sigma u}\right]_i^t \quad (16)$$

Here i indicate the intern static nodal variables and c the static nodal variable of contour.

2. VIRTUAL CRACK EXTENSION METHOD

The virtual crack extension method proposed by Parks [21] can calculate the energy release rate as:

$$G = -\frac{dU}{da} = -\frac{1}{2}\sum_{i=1}^{ne} \{u\}_i^t \frac{\Delta K_i}{\Delta a} \{u\}_i \quad (17)$$

where U is the potential energy of the system, a is the length of a crack, Δa is the length of the virtual crack extension, and ΔK_i and $\{u\}_i$ are the difference of the stiffness matrixes and the nodal displacements vectors of the elements i, surrounding a crack tip at the virtual at the virtual crack extension, respectively.

3. NUMERICAL EXAMPLES

3.1. Example 1.

In order to evaluate the validity and the credibility of the present element, a study of the convergence on a cantilever beam in flexion is carried out. A cantilever beam, with dimensions and loading as shown in figure 2, is subjected to two types of loading. The first type of loading corresponds to a uniform distribution of transversal load distributed on the end of the beam by respecting energy equivalence.

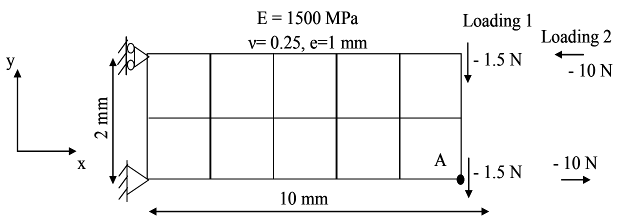

Figure 2 : Test of cantilever beam – Mesh

The second type of loading corresponds to a pure test of flexion. This problem has been solved by the present mixed element (for various meshes) to compare our result with the literature result, so as to gain additional confidence in the workability of the present element. Table 1 presents a comparison of deflections at the point A.

Table 1 : Comparison of deflection in a cantilever problem solved by various elements

Element type	Number of degrees of freedom	Deflection at point A (mm)	
		Loading 1	Loading 2
Displacement (4 nodes)	728	0.961	0.922
Reissner(4 nodes)	1431	1.023	0.998
Quad-1 Bichara [8]	498	1.029	1.000
Present mixed element	150	1.000	0.976
Classical theory	-	1.000	1.000
Timoshenko theory	-	1.030	1.000

Table 1 shows the good results obtained with the present mixed element compared with those of the analytical solution. Indeed, with a number of degrees of freedom definitely lower than those retained in the other comparative elements, the excellent results are obtained.

To see the convergence rapidity of the deflection at point A, several meshes are used. All the results obtained are reported in table 2 according to the number of elements and degrees of freedom.

Table 2 : Deflection in a cantilever problem solved by various meshes

Number of elements	Number of degrees of freedom	Deflection at point A (mm)	
		Loading 1	Loading 2
4	30	0.501	0.577
10	66	0.913	0.901
20	126	0.999	0.974
22	138	0.999	0.975
24	150	1.000	0.976

Figure 3 represents the convergence of the deflection. It is noted that the interface mixed element converges very quickly for a number relatively low of degrees of freedom.

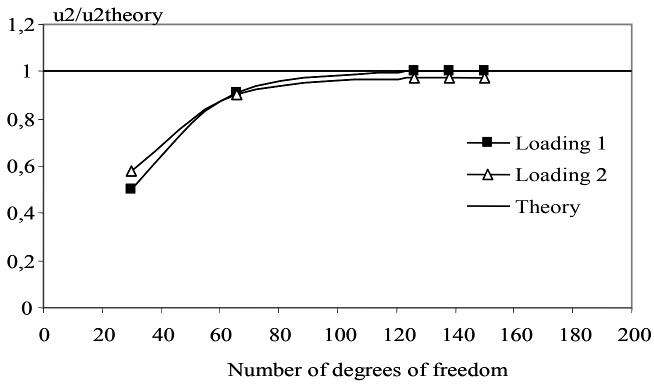

Figure 3 : Convergence of deflection in a cantilever beam

3.2. Example 2.

Simply supported sandwich beam has been considered. This beam presents three isotropic layers and presenting coherent interfaces. A sandwich beam, with dimensions and loading as shown in figure 4, is subjected to uniform load and the interest is primarily centered on the study of transverse shear stresses and the deflection.

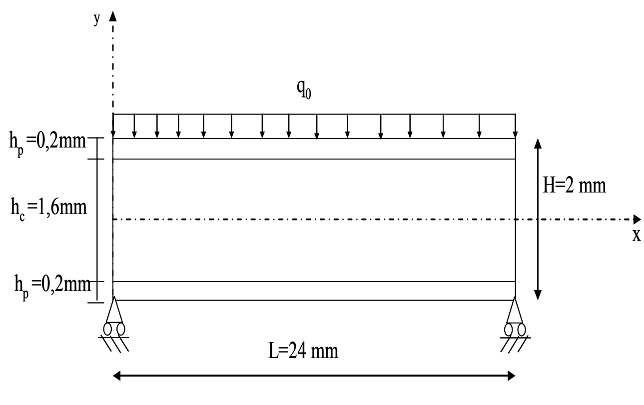

Figure 4 : Sandwich beam analyzed

The material properties of the face sheet and core material are :
- face sheet material (aluminium) (resin epoxy) :

E_p = 70000 MPa, ν_p=0,34
- core material :

E_c = 3400 MPa, ν_c=0,34

To see the convergence rapidity of the transverse shear and deflection several meshes are used. Results obtained trough the present mixed element for various numbers of degrees of freedom are tabulated in tables 3 and 4 where they have been compared with the elastic solutions given by Pagano [22]. Table 3 shows the deflection values obtained at x=L/2 according to the number of degrees of freedom. Variation of transverse shear at x = L/4 has been presented in table 4. It can be seen that the results from the present mixed element are in very good agreement with the elasticity solution [22].

Table 3 : Deflection in a sandwich beam at x=L/2 solved by various meshes

Element type	Number of degree of freedom	Deflection u_2 (mm)
Present mixed element	32	-0.105
	98	-0.200
	402	-0.209
Pagano [22]	-	-0.208

Table 4 : Transverse shear in a sandwich beam at x=L/4 solved by various meshes

Element type	Number of degree of freedom	(MPa) σ_{12} Transverse shear		
		y = - h_c/2	y = 0	y = h_c/2
Present mixed element	32	-3.026	-	-3.001
	98	-3.135	-	-3.258
	402	-3.197	-3.314	-3.184
Pagano [22]	-	-3.159	-3.431	-3.158

Figures 5 and 6 show the variation of the deflection and the transverse shear respectively with the number of degrees of freedom. It appears that the mixed interface element converges very quickly for a number relatively low of degrees of freedom.

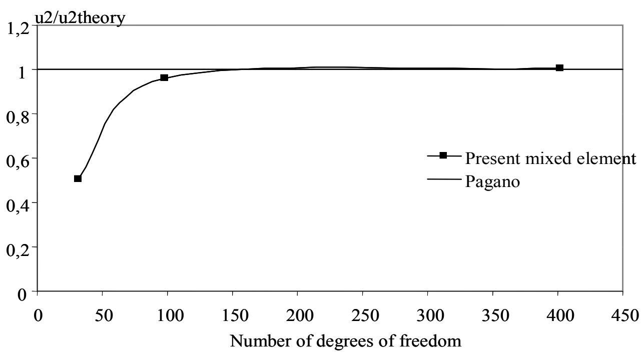

Figure 5 : Convergence of deflection at x = L/2

3.3. Example 3.

In this example, we analyzed a dissimilar square plate with a center crack of length 2a = 2mm in the interface plan between two isotropic materials subjected to tensile stress in the two directions as shown in figure 7.

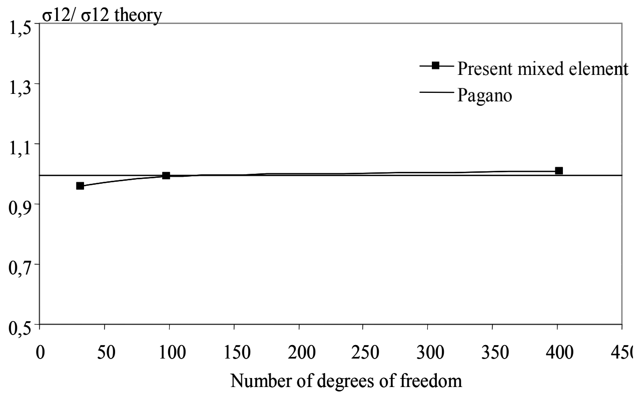

Figure 6 : Convergence of transverse shear at x=L/4 (y =- h_c/2)

We include the same geometrical and mechanical data of the reference [23].

In this example, the present element is associated to the virtual crack extension method to evaluate the energy release rate G. During numerical calculation, the choice of the crack length variation Δa is very important. To see the influence of this variation on the precision of calculation, we considered only one mesh with 50 elements and 286 degrees of freedom and we varied the extension in the interval $\frac{\Delta a}{a} = \frac{1}{10} \div \frac{1}{500}$.

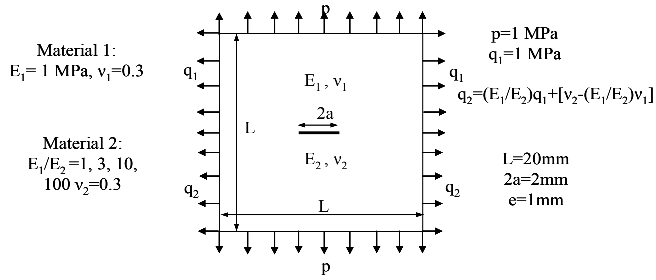

Figure 7 : A center crack in a dissimilar square plate

Results obtained with present interface element are compared with the values of the analytical solution [24] and the values of the numerical modelling of Lin and Mar [23]. These authors gave like results of their studies the stress intensity factors K_I and K_{II} from which we evaluated the energy release rate. Table 5 gives the values obtained according to E_1/ E_2.

Table 5 : Energy release rate of center interface crack between dissimilar materials

E_1/ E_2	Energy release rate G(N/mm)		
	Rice and Sih [24]	Lin and Mar [23]	Present mixed element
1	3.14	3.20	2.43
3	6.17	6.28	6.28
10	16.43	16.67	17.36
100	144.20	144.60	149.23

The results obtained confirm the importance of the choice of the extension and the validation of the present element for the cracked structure.

CONCLUSION

The mixed finite element method is used to derive a special interface element. The mixed variational formulation proves to be a very accurate method of numerical analysis for the evaluation of displacements and stresses of boundary value problem.

The present mixed element was built in order to answer as well as possible the conditions of continuity of displacement and stress vectors in the coherent part, and of discontinuity of displacements and effect edge on the cracked part. In the formulation of this element, we used Reissner's mixed variational principle to build the parent element. The mixed interface finite element is obtained by successively exploiting the technique of relocalisation and the static condensation procedure. The formulation starting from a parent element in a natural plane present the enormous advantage of modelling different types of cracks with various orientations. This interface element was associated with the virtual crack extension method to evaluate the energy release rates using only one meshing by finite elements. Economy of analysis is achieved when the present elements are included with a relatively smaller mesh and present sufficiently accurate results. The accuracy of the element has been evaluated by comparing the numerical solution with an available analytical solution or numerical ones obtained from others finite elements. Results obtained from the present mixed interface element have been shown to be in good agreement with the analytical solutions.

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