FLEXURAL VIBRATION OF A MULTI-DISK ROTOR WITH DIFFERENT BOUNDARY CONDITIONS

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Résumé

Dans cet article, les vibrations de flexion d'un rotor multi-disques sont considérées. L'analyse est effectuée pour le rotor avec les conditions aux limites appuyée-appuyée et appuyée-libre.

Celle-ci est faite dans le cas dissymétrique non amorti où les vitesses critiques et les réponses aux forces synchrones et asynchrones sont déterminées et comparées.

Mots clés *: Vibrations, rotor dynamiques, rotor flexible, conditions aux limites, vitesses critiques.*

Abstract

In this paper the flexural vibrations of a multi-disk rotor is considered. The analysis is made for the rotor when it is simply-simply supported and when it is free-simply supported.

This is done for the asymmetric undamped case where critical speeds and responses to synchronous and asynchronous forces are determined and compared.

Key words *: Vibrations, rotordynamics, flexible rotor, boundary conditions, critical speed.*

ملخص

في هذا المنشور تعتبر اهتزازات الالتواء لدوار متعدد الأقراص. يجرى التحليل للدوار عندما يكون لديه دعم بسيط-بسيط وعندما يكون لديه دعم بسيط-حر.

هذا يتم في الحالة غير المخمدة و غير المتناظرة حيث تحدد و تقارن السرعات الحرجة و الاستجابات للقوات المتزامنة وغير المتزامنة.

كلمات مفتاحيه *: الاهتزازات، ديناميكا الدوارات، الدوار المرن، الشروط الحدودية، السرعة الحرجة*

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1. INTRODUCTION

Rotating machinery, such as turbines, pumps, generators and fans, play an important role in many different industries where they are considered among the masterpieces in the mechanisms [1,2]. Unfortunately, they are sources of vibrations that involve the phenomena of the fatigue of their materials as well as some bad comfort qualities, in addition to the resonance phenomenon that leads to disasters if it is not avoided. To ensure a good running it is, therefore, necessary to get a precise knowledge of the vibratory behavior of the rotating parts. The main point of the problem lies in the determination of the critical speeds.

In a previous work [3,4], the study on the vibration dynamic behavior of a flexible mono rotor in the case where it is symmetric was considered. The model chosen was of Lalanne and Ferraris [5] with different boundary conditions. The obtained results led to markedly different critical speeds and modes of vibration. In this work the same method has been followed but in an asymmetric case and for a rotor with different number of discs.

2. EQUATIONS OF MOTION

The model of rotor considered in this work is represented in figure 1. The study of its flexural vibrations is made for the cases where it is simply-simply supported and free-simply supported to see the effect of the boundary conditions on the modes of vibration and the critical rotating speeds. For a better comparison, the work is made for the rotor when it is with just with one disc, with two discs and with three discs.

The caracteristics of the rotor, with the positions of the discs and bearings taken from the origin the inertial frame, are given as folows:

Shaft: Length L=1.3 m, cross section radius r= 0.05m, density ρ=7800kg/m^3, Young's modulus E=2.10^{11}N/m^2 and Poisson's coefficient υ=0.3.
Disc1: Inner radius r=0.05 m, outer radius r_1=0.12 m, thickness h_1= 0.05m, density ρ=7800kg/m^3 and position l_2=0.45 m.
Disc2: Inner radius r=0.05 m, outer radius r_2=0.20 m, thickness h_2=0.05 m, density ρ=7800kg/m^3 and position l_3=0.65 m.
Disc3: Inner radius r=0.05 m, outer radius r_3=0.20 m, thickness h_3=0.06 m, density ρ=7800kg/m^3 and position l_4= 0.85m.
Bearing (1): Position l_1=0.2 m, k_{xx}=5 10^7 N/m and k_{zz}=7 10^7 N/m.
Bearing (2): Position l_5=1.1 m, k_{xx}=5 10^7 N/m and k_{zz}=7 10^7 N/m.
Mass imbalance (1): m_{b1} = 8.8.10^{-5} kg and distance from the shaft axis d_1=0.12m.
Mass imbalance (2): $m_{b2(3)}$= 3.34.10^{-4} kg and distance from the shaft axis d_2=0. 20m.
From these data we obtain the physical quantities:
Shaft: Cross-section s=7.85.10^{-3}m^2 and diametral moment of inertia I= 4.906.10^{-6}m^4.
Disc1: Mass M_{d1}=14.57kg and moments of inertia I_{dx1}= I_{dz1}=0.06459kg m^2, I_{dy1}=0.123kgm^2.
Disc2: Mass M_{d2}=45.92kg and moments of inertia I_{dx2}= I_{dz2}= 0.497kgm^2, I_{dy2}=0.9758kgm^2.
Disc3: Mass M_{d3}=55.107kg and moments of inertia I_{dx3}= I_{dz3}= 0.602kgm^2, I_{dy3}=1.171kgm^2.

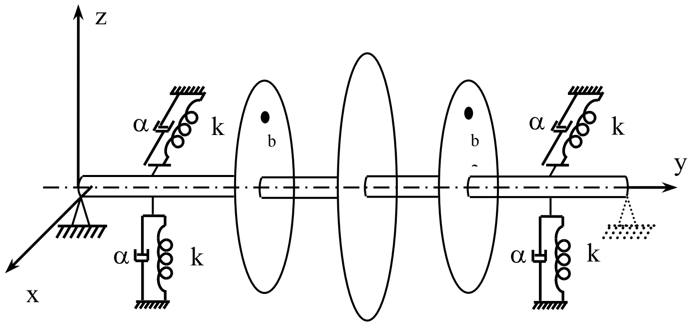

Figure 1 : Considered Rotating Model

The analysis of the flexural vibrations of the rotor is carried out by modeling based on the Rayleigh-Ritz method which is characterized by the substitution by approximation functions of the displacements u and w in the x and z directions respectively:

$$u(y,t) = f(y)q_1(t)=f(y)\,q_1 \tag{1}$$

and

$$w(y,t)=f(y)q_2(t)=f(y)q_2 \tag{2}$$

where q_1 and q_2 are the generalized independent coordinates and f(y) is the displacement function which is taken for the first mode of a beam in flexion with a constant cross-section. It is given by:

For simply-simply supported case:

$$f(y)=B\left[\sin\beta_n y\right] \tag{3}$$

where : $\beta_n L = \pi$.

For free-simply supported case,

$$f(y)=B\left[\sin\beta_n y+\alpha_n\, sh\beta_n y\right] \tag{4}$$

where: $\alpha_n = \dfrac{\sin\beta_n L}{sh\beta_n L}$ and $\beta_n L = 3.9266$.

B is a constant (taken equal to 1).

Consequently the expressions of the kinetic and stain energies (T and U) can be obtained.

The total kinetic energy of the system is:

$$T = T_s + T_d + T_b \tag{5}$$

Where T_s, T_d and T_b are the kinetic energy of the shaft, the masses unbalance and the discs respectively and which are given by [3] :

$$T_s = \frac{1}{2}\rho s(\dot{q}_1^2 + \dot{q}_2^2)\int_0^L f^2(y)dy \tag{6}$$

$$+\frac{1}{2}\rho I(\dot{q}_1^2 + \dot{q}_2^2)\int_0^L g^2(y)dy - 2\rho I\Omega\dot{q}_1 q_2\int_0^L g^2 y dy$$

$$T_d = T_{d1}+T_{d2}+T_{d3} \tag{7}$$

with :

$$T_{d1} = T_{d1}^{tra} + T_{d1}^{rot} = \frac{1}{2}M_{d1}f^2(l_2)(\dot{q}_1^2+\dot{q}_2^2)$$

$$+\frac{1}{2}Id_{x1}g^2(l_2)(\dot{q}_1^2+\dot{q}_2^2)+\frac{1}{2}Id_{y1}\Omega^2 - Id_{y1}g^2(l_2)\dot{q}_1 q_2\Omega$$

$$T_{d2} = T_{d2}^{tra} + T_{d2}^{rot} = \frac{1}{2}M_{d2}f^2(l_3)(\dot{q}_1^2+\dot{q}_2^2)$$

$$+\frac{1}{2}Id_{x2}g^2(l_3)(\dot{q}_1^2+\dot{q}_2^2)+\frac{1}{2}Id_{y2}\Omega^2 - Id_{y2}g^2(l_3)\dot{q}_1 q_2\Omega$$

and

$$T_{d3} = T_{d3}^{tra} + T_{d3}^{rot} = \frac{1}{2}M_{d3}f^2(l_4)(\dot{q}_1^2+\dot{q}_2^2)$$

$$+\frac{1}{2}Id_{x3}g^2(l_4)(\dot{q}_1^2+\dot{q}_2^2)+\frac{1}{2}Id_{y3}\Omega^2 - Id_{y3}g^2(l_4)\dot{q}_1 q_2\Omega$$

$$T_b = T_{b1}+T_{b2} \tag{8}$$

with

$$T_{b1} = m_{b1}d_1\Omega f(l_2)(\dot{q}_1\cos\Omega t - \dot{q}_2\sin\Omega t)$$

and

$$T_{b2} = m_{b2}d_2\Omega f(l_4)(\dot{q}_1\cos\Omega t - \dot{q}_2\sin\Omega t)$$

The total strain energy is that of the shaft and it is given by:

$$Ua = \frac{EI}{2}(q_1^2+q_2^2)\int_0^L h^2(y)dy \tag{9}$$

where: $h(y) = \dfrac{d^2 f(y)}{dy^2}$

The total virtual work due to the stiffness of the bearings is:

$$\delta W = -k_{xx}f^2(l_1)q_1\delta q_1 - k_{zz}f^2(l_1)q_2\delta q_2 - k_{xx}f^2(l_5)q_1\delta q_1 - k_{zz}f^2(l_5)q_2\delta q_2 \tag{10}$$

Following the procedure made by Lallane and Ferrari, the equations of motion are deducted using Lagrange equations for the dissymmetric model.

In the free-simply supported case,

- Rotor with one disc:

$$\begin{cases} 52.7006\ddot{q}_1 - 0.8269\,\Omega\dot{q}_2 + 86.216\,10^6 q_1 = 9.789\,10^{-6}\Omega^2\sin\Omega t \\ 52.7006\ddot{q}_2 + 0.8269\,\Omega\dot{q}_1 + 99.4867\,10^6 q_2 = 9.789\,10^6\Omega^2\cos\Omega t \end{cases} \tag{11}$$

- Rotor with two discs:

$$\begin{cases} 85.14\ddot{q}_1 - 2.91\,\Omega\dot{q}_2 + 86.216\,10^6 q_1 = 9.789\,10^{-6}\Omega^2\sin\Omega t \\ 85.14\ddot{q}_2 + 2.91\,\Omega\dot{q}_1 + 99.4867\,10^6 q_2 = 9.789\,10^{-6}\Omega^2\cos\Omega t \end{cases} \tag{12}$$

- Rotor with three discs:

$$\begin{cases} 98.126\ddot{q}_1 - 14.076\,\Omega\dot{q}_2 + 86.216\,10^6 q_1 = 3.7199\,10^{-5}\Omega^2\sin\Omega t \\ 98.126\ddot{q}_2 + 14.076\Omega\dot{q}_1 + 99.4867\,10^6 q_2 = 3.7199\,10^{-5}\Omega^2\cos\Omega t \end{cases} \tag{13}$$

In the simply-simply supported case,

- Rotor with one disc:

$$\begin{cases} 46.754\ddot{q}_1 - 0.4793\,\Omega\dot{q}_2 + 43.358\,10^6 q_1 = 9.3477\,10^{-6}\Omega^2\sin\Omega t \\ 46.754\ddot{q}_2 + 0.4793\,\Omega\dot{q}_1 + 52.015\,10^6 q_2 = 9.3477\,10^{-6}\Omega^2\cos\Omega t \end{cases} \tag{14}$$

- Rotor with two discs:

$$\begin{cases} 111.0564\ddot{q}_1 - 35.6207\,\Omega\dot{q}_2 + 43.358\,10^6 q_1 = 9.3477\,10^{-6}\Omega^2\sin\Omega t \\ 111.0564\ddot{q}_2 + 35.6207\,\Omega\dot{q}_1 + 52.015\,10^6 q_2 = 9.3477\,10^{-6}\Omega^2\cos\Omega t \end{cases} \tag{15}$$

- Rotor with three discs:

$$\begin{cases} 155.0624\ddot{q}_1 - 37.0897\,\Omega\dot{q}_2 + 43.358\,10^6 q_1 = 1.00174\,10^{-5}\Omega^2\sin\Omega t \\ 155.0624\ddot{q}_2 + 37.0897\,\Omega\dot{q}_1 + 52.015\,10^6 q_2 = 1.00174\,10^{-5}\Omega^2\cos\Omega t \end{cases} \tag{16}$$

Any of the above systems of equations can be put in the form

$$\begin{cases} m\ddot{q}_1 - a\Omega\dot{q}_2 + k_1 q_1 = C\Omega^2\sin\Omega t \\ m\ddot{q}_2 + a\Omega\dot{q}_1 + k_2 q_2 = C\Omega^2\cos\Omega t \end{cases} \tag{17}$$

or in the matrix form:

$$\begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix}\begin{bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{bmatrix} + \Omega\begin{bmatrix} 0 & -a \\ a & 0 \end{bmatrix}\begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} + \begin{bmatrix} k_1 & 0 \\ 0 & k_2 \end{bmatrix}\begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = C\Omega^2\begin{bmatrix} \sin\Omega t \\ \cos\Omega t \end{bmatrix} \tag{18}$$

3. NATURAL FREQUENCIES AND EIGENMODES

The natural frequencies are found by solving the homogeneous system of equations (17) i.e the system without second member. Since this one is linear the solutions have the following forms:

$$\begin{cases} q_{1h} = A_1\cos(\omega t + \phi_1) \\ q_{2h} = A_2\cos(\omega t + \phi_2) \end{cases} \tag{19}$$

These can be transformed in a complex form as:

$$\begin{cases} \underline{q}_{1h} = \underline{A}_1\exp j\omega t \\ \underline{q}_{2h} = \underline{A}_2\exp j\omega t \end{cases} \tag{20}$$

with $\underline{A}_1 = A_1\exp j\phi_1$ and $\underline{A}_2 = A_2\exp j\phi_2$.

By substitution in the homogenous complex system we get:

$$\begin{cases} m\ddot{\underline{q}}_{1h} - a\Omega\dot{\underline{q}}_{2h} + k_1\underline{q}_{1h} = 0 \\ m\ddot{\underline{q}}_{2h} + a\Omega\dot{\underline{q}}_{1h} + k_2\underline{q}_{2h} = 0 \end{cases} \quad (21)$$

Or in the matrix form

$$\begin{bmatrix} k_1 - m\omega^2 & -ja\Omega a \\ ja\Omega a & k_2 - m\omega^2 \end{bmatrix}\begin{bmatrix} \underline{A}_1 \\ \underline{A}_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad (22)$$

This represents a linear system of equations with two unknown $\underline{A}_1$ And $\underline{A}_2$ with a parameter ω which is discussed for the non-trivial case that corresponds to determinant =0.

That is:

$$\det = \begin{bmatrix} k_1 - m\omega^2 & -ja\Omega a \\ ja\Omega a & k_2 - m\omega^2 \end{bmatrix} = 0 \quad (23)$$

or : $m^2\omega^4 - (k_1 m + k_2 m + a^2\Omega^2)\,\omega^2 + k_1 k_2 = 0$ (24)

We remark that when $\Omega = 0$ (rotor at rest), the solutions of the (24) are equal to:

$$\omega_{10} = \sqrt{k_1/m} \quad (25)$$

and :

$$\omega_{20} = \sqrt{k_2/m} \quad (26)$$

On the other hand when $\Omega \neq 0$ (rotating rotor), the bi-squared equation has a positive discriminant which means that it has two positive real values for ω given by:

$$\omega_1 = \sqrt{\frac{\omega_{10}^2}{2} + \frac{\omega_{20}^2}{2} + \frac{a^2\Omega^2}{2m^2} - \sqrt{\left(\frac{\omega_{10}^2}{2} + \frac{\omega_{20}^2}{2} + \frac{a^2\Omega^2}{2m^2}\right)^2 - \omega_{10}^2\omega_{20}^2}} \quad (27)$$

and

$$\omega_2 = \sqrt{\frac{\omega_{10}^2}{2} + \frac{\omega_{20}^2}{2} + \frac{a^2\Omega^2}{2m^2} + \sqrt{\left(\frac{\omega_{10}^2}{2} + \frac{\omega_{20}^2}{2} + \frac{a^2\Omega^2}{2m^2}\right)^2 - \omega_{10}^2\omega_{20}^2}} \quad (28)$$

We can find that:

$$\omega_1 < \omega_{10} < \omega_{20} < \omega_2 \quad (29)$$

There are then two modes of vibration for each of the generalized coordinates q_{1h} and q_{2h} corresponding to the two values ω_1 and ω_2.

i) For the first mode (**$\omega = \omega_1$**), we have:

$$\frac{\underline{A}_{11}}{\underline{A}_{21}} = \frac{ja\Omega\omega_1}{k_1 - m\omega_1^2} = \frac{ja\Omega\omega_1}{m(\omega_{10}^2 - \omega_1^2)} = j \quad (30)$$

That gives: $A_{11} = A_{21}$ and $\varphi_{21} = \varphi_{11} - \pi/2$

Hence :

$$q_{1h1} = A_{11}\cos(\omega_1 t + \phi_{11})$$

and

$$q_{2h1} = A_{21}\cos(\omega_1 t + \phi_{21}) = A_{11}\sin(\omega_1 t + \phi_{11}) \quad (31)$$

Then,

$$u(y,t) = q_{1h1}f(y) = A_{11}\cos(\omega_1 t + \phi_{11})\left[\sin\beta_n y + \alpha_n \mathrm{sh}\beta_n y\right] \quad (32)$$

and

$$w(y,t) = q_{2h1}f(y) = A_{11}\sin(\omega_1 t + \phi_{11})\left[\sin\beta_n y + \alpha_n \mathrm{sh}\beta_n y\right] \quad (33)$$

ii) For the second mode (**$\omega = \omega_2$**), we have:

$$\frac{\underline{A}_{12}}{\underline{A}_{22}} = \frac{ja\Omega\omega_2}{k_1 - m\omega_2^2} = \frac{ja\Omega\omega_2}{m(\omega_{01}^2 - \omega_2^2)} = -j \quad (34)$$

That gives: $A_{12}=A_{22}$ et $\varphi_{22}= \varphi_{12} + \pi/2$

Hence:

$$q_{1h2} = A_{12}\cos(\omega_2 t + \phi_{12})$$

and

$$q_{2h2} = A_{22}\cos(\omega_2 t + \phi_{22}) = -A_{12}\sin(\omega_2 t + \phi_{12}) \quad (35)$$

Then,

$$u(y,t) = q_{1h2}f(y) = A_{12}\cos(\omega_2 t + \phi_{12})\left[\sin\beta_n y + \alpha_n \mathrm{sh}\beta_n y\right] \quad (36)$$

And

$$w(y,t) = q_{2h2}f(y) = A_{22}\cos(\omega_2 t + \phi_{22}\left[\sin\beta_n y + \alpha_n \mathrm{sh}\beta_n y\right] \quad (37)$$

As A_1 is different from A_2, the orbits described by the rotor are ellipses with an inverse precession for the first mode and a direct precession for second one.

4. CAMPBELL DIAGRAM

The characteristic equation for each considered case is obtained from the corresponding system of equations (11) to (16). It allows having the frequencies at rest and the frequencies in rotation.

In Campbell Diagram (Figure 2 and Figure 3), the functions $\omega_1=\omega_1(\Omega)$ et $\omega_2 = \omega_2(\Omega)$ are represented cut by the straight lines $\omega=\Omega$ (the synchronous case) and $\omega = s\Omega$ (an asynchronous case with s= 0.5) to get the intersection points A and B for the first and, C and D for the second.

The values of the frequencies corresponding for these points are obtained using $\omega = s\Omega$ in (24), that is:

$$s^2(s^2m^2 - a^2)\Omega^4 - m(k_1 + k_2)s^2\Omega^2 + k_1k_2 = 0 \quad (38)$$

From this we find the critical value Ω_c for the cases s=1 and s=0.5.

In the simply-simply supported case,

- Rotor with one disc :

$$\omega^4 - (20.398.10^5 + 0.2297\Omega^2)\omega^2 + 2.255\,10^{15} = 0 \quad (39)$$

$$\omega_{10} = \sqrt{k_1/m} = 962.997 \text{ rd/s}$$

$$\text{and} \quad \omega_{20} = \sqrt{k_1/m} = 1057.763 \text{ rd/s} \quad (40)$$

$$\omega_1 = \sqrt{10.199\,10^5 + 2.456\,10^{-3}\Omega^2 - \sqrt{(10.199\,10^5 + 2.456\,10^{-3}\Omega^2)^2 - 1.0317\,10^{12}}}$$

$$\omega_2 = \sqrt{10.199.10^5 + 2.456\,10^{-3}\Omega^2 + \sqrt{\left(10.199\,10^5 + 2.456\,10^{-3}\Omega^2\right)^2 - 1.0317\,10^{12}}} \quad (40)$$

A: Ω_{c1}=958 rd/s ; B: Ω_{c2}=050.954 rd/s,
C: Ω_{c1}=1812.974 rd/s ; D: Ω_{c2}=2192.32 rd/s.

- Rotor with two discs :

$$\omega^4 - (8.587\,10^5 + 12.68\,10^2\Omega^2)\omega^2 + 2.255\ 10^{15} = 0 \quad (42)$$

$$\omega_{10} = \sqrt{k_1 / m} = 624.831 \text{ rd/s and } \omega_{20} = \sqrt{k_1 / m} = 684.372 \text{ rd/s} \quad (43)$$

$$\omega_1 = \sqrt{4.2938\,10^5 + 5.1438\,10^{-2}\Omega^2 - \sqrt{\left(4.2938\,10^5 + 5.1438\,10^{-2}\Omega^2\right)^2 - 1.828\,10^{11}}}$$

$$\omega_2 = \sqrt{4.2938\,10^5 + 5.1438\,10^{-2}\Omega^2 + \sqrt{\left(4.2938\,10^5 + 5.1438\,10^{-2}\Omega^2\right)^2 - 1.828\,10^{11}}} \quad (44)$$

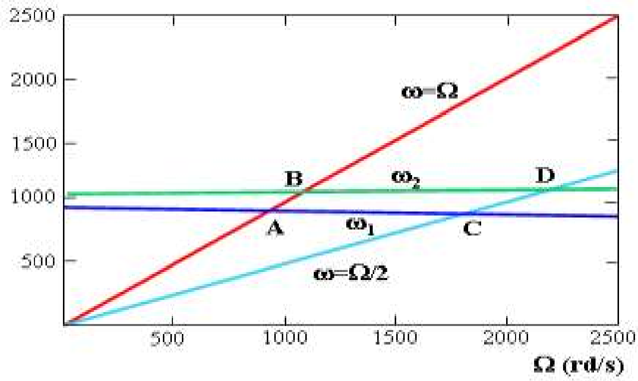

Figure 2(a) : Rotor with one disc, simply-simply supported case

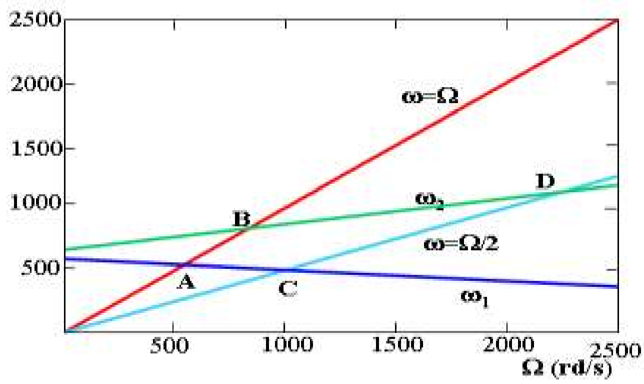

Figure 2(b) : Rotor with two discs, simply-simply supported case

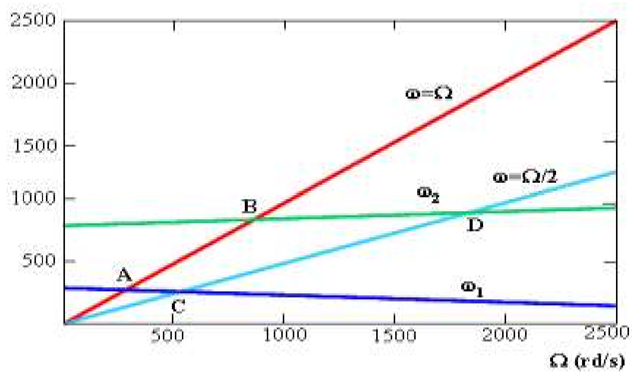

Figure 2(c) : Rotor with three discs, simply-simply supported case

A: Ω_{c1} = 565.356 rd/s ; B: Ω_{c2}=798.5135 rd/s,
C: Ω_{c1}=1017. 45 rd/s ; D: Ω_{c2}=2191.33rd/s.

- Rotor with three discs:

$$\omega^4 - (61.506\,10^4 + 13.756\,10^2\Omega^2)\omega^2 + 2.255.10^{15} = 0 \quad (45)$$

$$\omega_{10} = \sqrt{k_1 / m} = 528.787 \text{ rd/s and } \omega_{20} = \sqrt{k_1 / m} = 579.176 \text{ rd/s} \quad (46)$$

$$\omega_1 = \sqrt{30.7531\,10^4 + 2.86\,10^{-2}\Omega^2 - \sqrt{\left(30.7531\,10^4 + 2.86\,10^{-2}\Omega^2\right)^2 - 2.3449\,10^{10}}}$$

$$\omega_2 = \sqrt{30.7531\,10^4 + 2.86\,10^{-2}\Omega^2 + \sqrt{\left(30.7531\,10^4 + 2.86.10^{-2}\Omega^2\right)^2 - 2.3449\,10^{10}}} \quad (47)$$

A: Ω_{c1}=492.917 rd/s ; B: Ω_{c2}=639.857 rd/s,
C: Ω_{c1}=906.314 rd/s ; D: Ω_{c2}=1772.77rd/s.

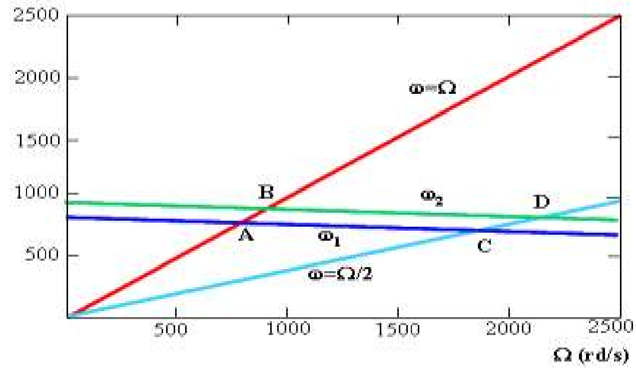

Figure 3(a) : Rotor with one disc, free-simply supported case

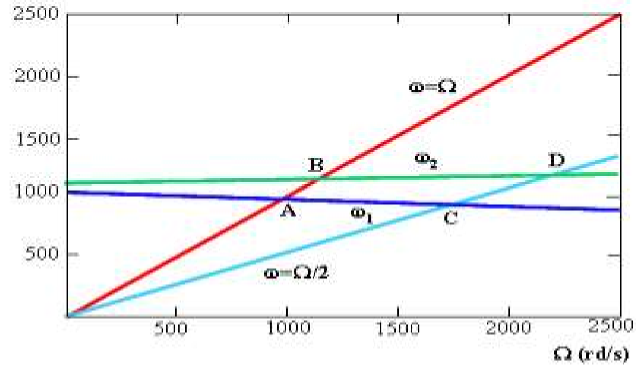

Figure 3(b) : Rotor with two discs, free-simply supported case

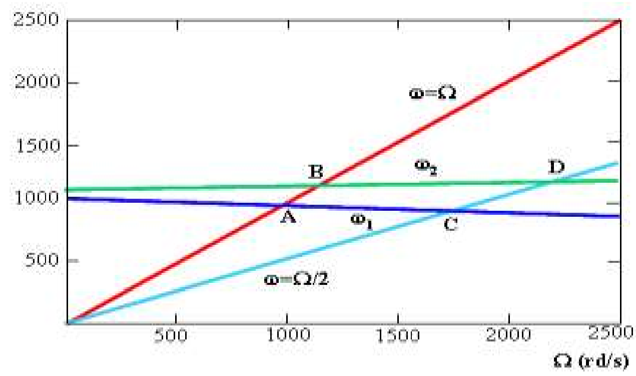

Figure 3(c) : Rotor with three discs, free-simply supported case

In the free-simply supported case,

- Rotor with one disc:

$$\omega^4 - (35.237\ 10^5 + 2.4619\ 10^{-4}\Omega^2)\omega^2 + 3.0883\ 10^{12} = 0 \quad (48)$$

$$\omega_{10} = \sqrt{k_1 / m} = 1279.0459 \text{ rd/s}$$

and $\omega_{20} = \sqrt{k_1 / m} = 1373.962 \text{ rd/s}$ (49)

$$\omega_1 = \sqrt{35.237\ 10^5 + 2.462\ 10^{-4} - \sqrt{(35.273\ 10^5 + 2.462\ 10^{-4}\Omega^2)^2 - 3.0883\ 10^{12}}}$$

$$\omega_2 = \sqrt{35.237\ 10^5 + 2.462\ 10^{-4} + \sqrt{(35.273\ 10^5 + 2.462\ 10^{-4}\Omega^2)^2 - 3.0883\ 10^{12}}} \quad (50)$$

A: Ω_{c1} =1278.036 rd/s ; B: Ω_{c2} =1375.2168 rd/s,
C: Ω_{c1} =2550.3034 rd/s ; D: Ω_{c2} =2757.667 rd/s.

- Rotor with two discs:

$$\omega^4 - (21.81\,10^5 + 1.168\ 10^{-3}\Omega^2)\omega^2 + 1.18327\ 10^{12} = 0 \quad (51)$$

$$\omega_{10} = \sqrt{k_1 / m} = 1006.299 \text{ rd/s}$$

and $\omega_{20} = \sqrt{k_1 / m} = 1080.975 \text{ rd/s}$ (52)

$$\omega_1 = \sqrt{10.9057\ 10^5 + 5.841\ 10^{-4} - \sqrt{(10.9057\ 10^5 + 5.841\ 10^{-4}\Omega^2)^2 - 1.18327\ 10^{12}}}$$

$$\omega_2 = \sqrt{10.9057\ 10^5 + 5.841\ 10^{-4} + \sqrt{(10.9057\ 10^5 + 5.841\ 10^{-4}\Omega^2)^2 - 1.18327\ 10^{12}}} \quad (53)$$

A: Ω_{c1} =1002. 69 rd/s ; B: Ω_{c2} =1085.49 rd/s,
C: Ω_{c1} =1987. 43 rd/s ; D: Ω_{c2} =2194.46 rd/s.

- Rotor with three discs:

$$\omega^4 - (21.81\,10^5 + 1.168\ 10^{-3}\Omega^2)\omega^2 + 1.18327\ 10^{12} = 0 \quad (54)$$

$$\omega_{10} = \sqrt{k_1 / m} = 1006.299 \text{ rd/s}$$

and $\omega_{20} = \sqrt{k_1 / m} = 1080.975 \text{ rd/s}$ (55)

$$\omega_1 = \sqrt{10.9057\ 10^5 + 5.841\ 10^{-4} - \sqrt{(10.9057\ 10^5 + 5.841\ 10^{-4}\Omega^2)^2 - 1.18327\ 10^{12}}}$$

$$\omega_2 = \sqrt{10.9057\ 10^5 + 5.841\ 10^{-4} + \sqrt{(10.9057\ 10^5 + 5.841\ 10^{-4}\Omega^2)^2 - 1.18327\ 10^{12}}} \quad (56)$$

A: Ω_{c1} =900. 89 rd/s ; B: Ω_{c2} =1058. 6 rd/s,
C: Ω_{c1} =1705. 27 rd/s; D: Ω_{c2} =2311.056 rd/s.

5. RESPONSE TO A SYNCHRONOUS FORCE

When considering the excitation force due to the mass-unbalance, the steady state solution which is the particular solution is found by solving the inhomogeneous system of equations (17) i.e the system with second member. For this one, the solutions of the system may have the form:

$$\begin{cases} q_{1p} = A_{e1}\cos(\Omega t + \phi_{e1}) \\ q_{2p} = A_{e2}\cos(\Omega t + \phi_{e2}) \end{cases} \quad (57)$$

They can be transformed in a complex form as:

$$\begin{cases} \underline{q}_{1p} = \underline{A}_{e1}\exp j\Omega t \\ \underline{q}_{2p} = \underline{A}_{e2}\exp j\Omega t \end{cases} \quad (58)$$

With: $\underline{A}_{e1} = A_{e1}\exp j\phi_{e1}$ and $\underline{A}_{e2} = A_{e2}\exp j\phi_{e2}$

Their introduction in the inhomogeneous complex system yields:

$$\begin{cases} m\underline{\ddot{q}}_{1p} - a\Omega\underline{\dot{q}}_{2p} + k_1\underline{q}_{1p} = C\Omega^2\exp j(\Omega t - \pi/2) \\ m\underline{\ddot{q}}_{2p} + a\Omega\underline{\dot{q}}_{1p} + k_2\underline{q}_{2p} = C\Omega^2\exp j(\Omega t) \end{cases} \quad (59)$$

or

$$\begin{bmatrix} k_1 - m\Omega^2 & -ja\Omega^2 \\ ja\Omega^2 & k_2 - m\Omega^2 \end{bmatrix}\begin{bmatrix} \underline{A}_{e1} \\ \underline{A}_{e2} \end{bmatrix} = \begin{bmatrix} C\Omega^2\exp(-j\pi/2) \\ C\Omega^2 \end{bmatrix} \quad (60)$$

This last system of equations represents a linear system with two unknowns $\underline{A}_{e1}$ et $\underline{A}_{e2}$ that depend on a parameter Ω. The determinant method gives:

$$\underline{A}_{e1} = \frac{\begin{vmatrix} C\Omega^2 e^{-j\pi/2} & -ja\Omega^2 \\ C\Omega^2 & k_2 - m\Omega^2 \end{vmatrix}}{\begin{vmatrix} k_1 - m\Omega^2 & -ja\Omega^2 \\ ja\Omega^2 & k_2 - m\Omega^2 \end{vmatrix}} = \frac{C\Omega^2(k_2 - m\Omega^2)e^{-j\pi/2} + jaC\Omega^4}{\left[(k_1 - m\Omega^2)(k_2 - m\Omega^2)\ - a^2\Omega^4\right]}$$

$$= \frac{-jC\Omega^2(k_2 - m\Omega^2 - a\Omega^2)}{\left[(k_1 - m\Omega^2)(k_2 - m\Omega^2)\ - a^2\Omega^4\right]} \quad (61)$$

$$\underline{A}_{e2} = \frac{\begin{vmatrix} k_1 - m\Omega^2 & C\Omega^2 e^{-j\pi/2} \\ ja\Omega^2 & C\Omega^2 \end{vmatrix}}{\begin{vmatrix} k_1 - m\Omega^2 & -ja\Omega^2 \\ ja\Omega^2 & k_2 - m\Omega^2 \end{vmatrix}} = \frac{C\Omega^2(k_1 - m\Omega^2) + jaC\Omega^4 e^{-j\pi/2}}{\left[(k_1 - m\Omega^2)(k_2 - m\Omega^2)\ - a^2\Omega^4\right]}$$

$$= \frac{C\Omega^2(k_1 - m\Omega^2 - a\Omega^2)}{\left[(k_1 - m\Omega^2)(k_2 - m\Omega^2)\ - a^2\Omega^4\right]} \quad (62)$$

From these relations, we obtain:

$$A_{e1} = \frac{\left[k_2 - (m+a)\Omega^2\right]C\Omega^2}{(k_1 - m\Omega^2)(k_2 - m\Omega^2) - a^2\Omega^4}$$

and (63)

$$A_{e2} = \frac{\left[k_1 - (m+a)\Omega^2\right]C\Omega^2}{(k_1 - m\Omega^2)(k_2 - m\Omega^2) - a^2\Omega^4}$$

Also $\varphi_{e1} = -\pi/2$ and $\varphi_{e2} = 0$

So we can write:

$q_{1p} = A_{e1}\sin\Omega t$ and $q_{2p} = A_{e2}\cos\Omega t$ (64)

As A_{e1} et A_{e2} are different, the orbits described by the rotor are ellipses. The expressions of the critical speeds come from the denominator of the equation (62) when it is equal to zero, that is:

$$(m^2 - a^2)\Omega^4 - m(k_1 + k_2)\Omega^2 + k_1 k_2 = 0 \quad (65)$$

We remark that this equation is similar to equation (38) for the case where s = 1. The two critical speeds of rotation correspond to the points A and B. However the sense of precession is given by the product of the amplitudes. Indeed, if $A_{e1}.A_{e2} > 0$ the precession is direct and if $A_{e1}.A_{e2} < 0$ the precession is reverse.

In order to know the sign of $A_{e1}.A_{e2}$, we have to know the sign of the following function:

$$f(\Omega^2) = \left[k_1 - (m+a)\Omega^2\right]\left[k_2 - (m+a)\Omega^2\right] \quad (66)$$

It is equal to zero for the two values of Ω:

$$\Omega_1 = \sqrt{k_1/(m+a)} \text{ and } \Omega_2 = \sqrt{k_2/(m+a)}$$

These allow having the sense of the precession for each value of the rotational speed (see figure 4).

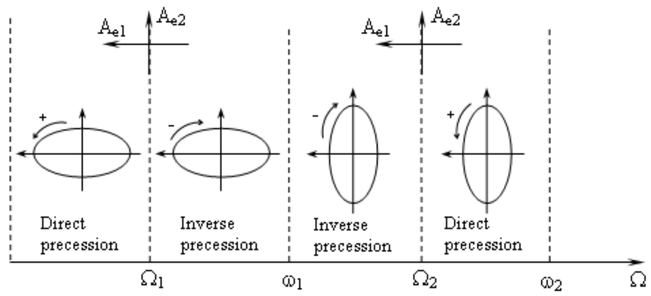

Figure 4 : Variation of the sense of the precession

For the different cases of the considered model of rotor defined by equations (11) to (16), we have:

In the simply-simply supported case,

- Rotor with one disc:

$$A_{e1} = \frac{\left[486{,}220 - 4{,}415.10^{-4}\Omega^2\right]\Omega^2}{2185{,}706\Omega^4 - 44{,}59.10^8\Omega^2 + 2{,}255.10^{15}},$$

$$A_{e2} = \frac{\left[405{,}297 - 4{,}415.10^{-4}\Omega^2\right]\Omega^2}{2185{,}706\Omega^4 - 44{,}59.10^8\Omega^2 + 2{,}255.10^{15}} \quad (67)$$

- Rotor with two discs:

$$A_{e1} = \frac{\left[486{,}220 - 1{,}371.10^{-3}\Omega^2\right]\Omega^2}{110{,}646.10^2\Omega^4 - 1{,}059.10^{10}\Omega^2 + 2{,}255.10^{15}},$$

$$A_{e2} = \frac{\left[405{,}297 - 1{,}371.10^{-3}\Omega^2\right]\Omega^2}{110{,}646.10^2\Omega^4 - 1{,}059.10^{10}\Omega^2 + 2{,}255.10^{15}} \quad (68)$$

- Rotor with three discs:

$$A_{e1} = \frac{\left[521{,}055 - 1{,}9248.10^{-3}\Omega^2\right]\Omega^2}{22{,}668.10^3\Omega^4 - 1{,}4788.10^{10}\Omega^2 + 2{,}255.10^{15}},$$

$$A_{e2} = \frac{\left[434{,}334 - 1{,}9248.10^{-3}\Omega^2\right]\Omega^2}{22{,}668.10^3\Omega^4 - 1{,}4788.10^{10}\Omega^2 + 2{,}255.10^{15}} \quad (69)$$

In the free-simply supported case,

- Rotor with one disc:

$$A_{e1} = \frac{\left[973{,}875 - 5{,}1786.10^{-4}\Omega^2\right]\Omega^2}{2703{,}81\Omega^4 - 96{,}5765.10^8\Omega^2 + 8{,}577.10^{15}},$$

$$A_{e2} = \frac{\left[843{,}968 - 5{,}1786.10^{-4}\Omega^2\right]\Omega^2}{2703{,}81\Omega^4 - 96{,}5765.10^8\Omega^2 + 8{,}577.10^{15}} \quad (70)$$

- Rotor with two discs:

$$A_{e1} = \frac{\left[973{,}875 - 8{,}619.10^{-4}\Omega^2\right]\Omega^2}{7240{,}352\Omega^4 - 1{,}581.10^{10}\Omega^2 + 8{,}577.10^{15}},$$

$$A_{e2} = \frac{\left[843{,}968 - 8{,}619.10^{-4}\Omega^2\right]\Omega^2}{7240{,}352\Omega^4 - 1{,}581.10^{10}\Omega^2 + 8{,}577.10^{15}} \quad (71)$$

- Rotor with three discs:

$$A_{e1} = \frac{\left[3700{,}8 - 4{,}1738.10^{-3}\Omega^2\right]\Omega^2}{9430{,}576\Omega^4 - 1{,}822.10^{10}\Omega^2 + 8{,}577.10^{15}},$$

$$A_{e2} = \frac{\left[3207{,}149 - 4{,}1738.10^{-3}\Omega^2\right]\Omega^2}{9430{,}576\Omega^4 - 1{,}822.10^{10}\Omega^2 + 8{,}577.10^{15}} \quad (72)$$

Ae_1 and Ae_2 are represented in absolute values on figure 5 and figure 6, where we see that the phenomenon of resonance occurs for two critical values unlike the symmetric case where it occurs for a single value. On the other hand, when $\Omega >> 0$ (very positive), the amplitudes A_{e1} et A_{e2} will be equal and tend to a constant value equals to: 9.8338 10^{-7}m.

6. REPONSE TO AN ASYNCHRONOUS FORCE

The rotor can also be excited by an asynchronous force during its operation. It is a force with a constant amplitude F_0 and speeds $s\Omega$ different from that of the rotor. If the force is applied in l_3 we have:

$$F_{q1} = F_0 f(l_3)\sin s\Omega t = F\sin s\Omega t$$

and $F_{q2} = F_0 f(l_3)\cos s\Omega t = F\cos s\Omega t$ (73)

The equations to be resolved will be :

$$\begin{cases} m\ddot{q}_1 - a\Omega\dot{q}_2 + k_1 q_1 = F\sin s\Omega t \\ m\ddot{q}_2 + a\Omega\dot{q}_1 + k_2 q_2 = F\cos s\Omega t \end{cases} \quad (74)$$

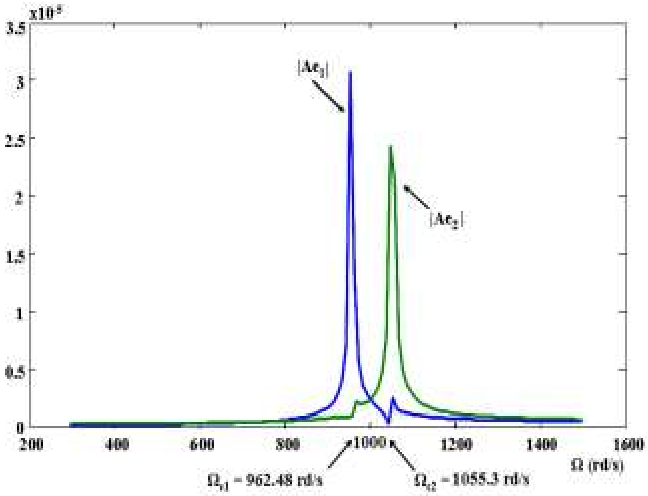

Figure 5(a) : Response to a synchronous force: rotor with one disc, simply-simply supported case

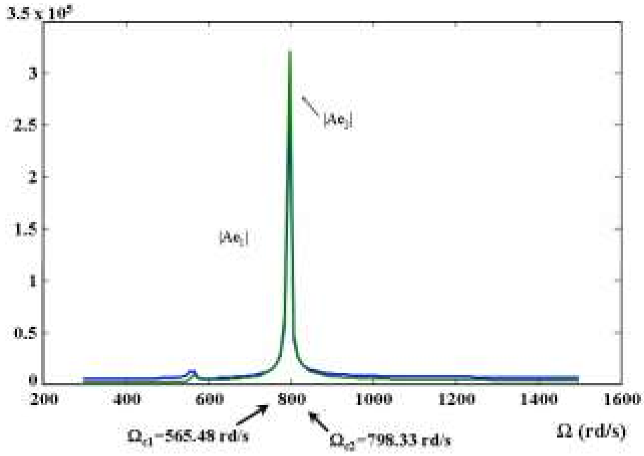

Figure 5(b) : Response to a synchronous force: rotor with two discs, simply-simply supported case

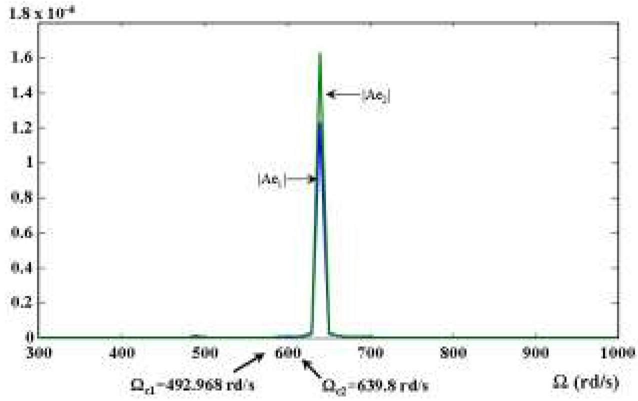

Figure 5(c) : Response to a synchronous force: rotor with three discs, simply-simply supported case

As the precedent case, the solutions are of the form:

$$A_{e1} = \frac{\left[k_2 - (ms^2 + as)\Omega^2\right]F}{s^2(s^2m^2 - a^2)\Omega^4 - ms^2(k_1 + k_2)\Omega^2 + k_1k_2}$$

And (75)

$$A_{e2} = \frac{\left[k_1 - (mh^2 + as)\Omega^2\right]F}{s^2(s^2m^2 - a^2)\Omega^4 - ms^2(k_1 + k_2)\Omega^2 + k_1k_2}$$

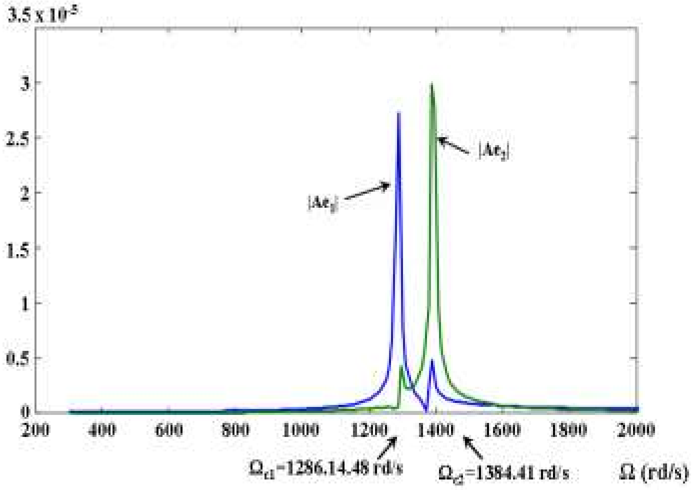

Figure 6(a) : Response to a synchronous force: rotor with one disc, free-simply supported case.

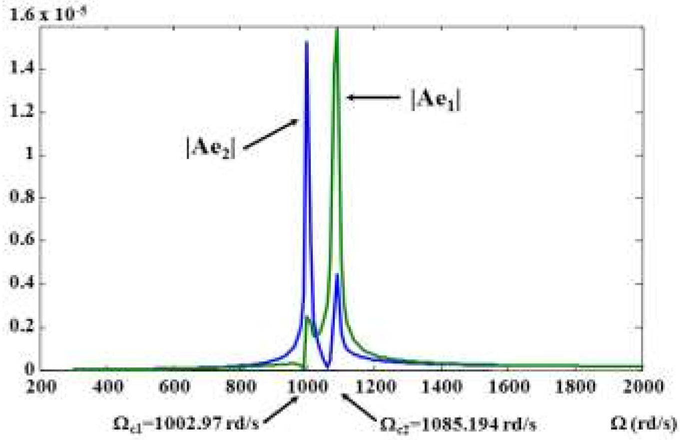

Figure 6(b) : Response to a synchronous force: rotor with two discs free-simply supported case.

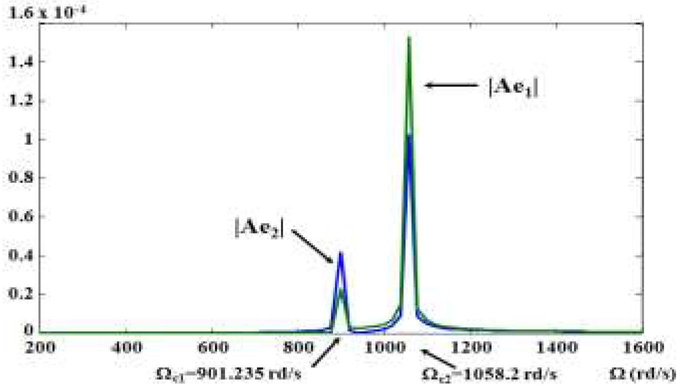

Figure 6(c) : Response to a synchronous force: rotor with three discs free-simply supported case.

The critical speeds are given by the same equation (38) and the orbits are described by ellipses.

$$\begin{cases} q_{1p} = A_{e1}\cos(s\Omega t + \phi_{e1}) \\ q_{2p} = A_{e2}\cos(s\Omega t + \phi_{e2}) \end{cases} \tag{76}$$

The calculations lead us to:

In the simply-simply supported case,

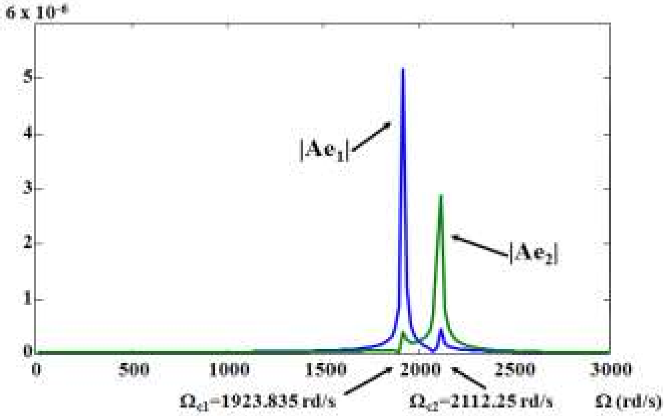

Figure 7(a) : Response to an asynchronous force: rotor with one disc simply-simply supported case.

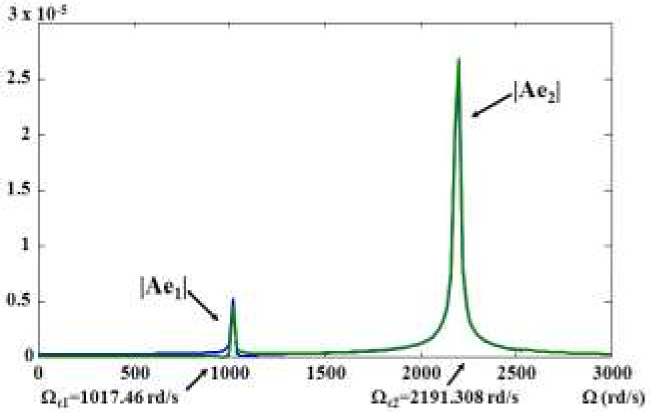

Figure 7(b) : Response to an asynchronous force: rotor with two discs simply-simply supported case.

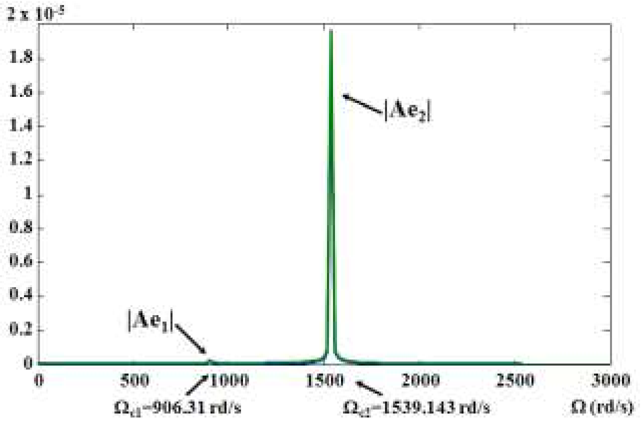

Figure 7(c) : Response to an asynchronous force: rotor with three discs simply-simply supported case.

- Rotor with one disc:

$$A_{e1} = \frac{(52{,}015.10^6 - 11{,}928\ \Omega^2)F}{136{,}5636\ \Omega^4 - 11{,}147.10^8\Omega^2 + 2{,}255.10^{15}},$$

$$A_{e2} = \frac{(43{,}358.10^6 - 11{,}928\ \Omega^2)F}{136{,}5636\ \ \Omega^4 - 11{,}147.10^8\Omega^2 + 2{,}255.10^{15}} \tag{77}$$

- Rotor with two discs:

$$A_{e1} = \frac{(52{,}015.10^6 - 45{,}574\ \Omega^2)F}{453{,}636\ \Omega^4 - 26{,}479.10^8\Omega^2 + 2{,}255.10^{15}},$$

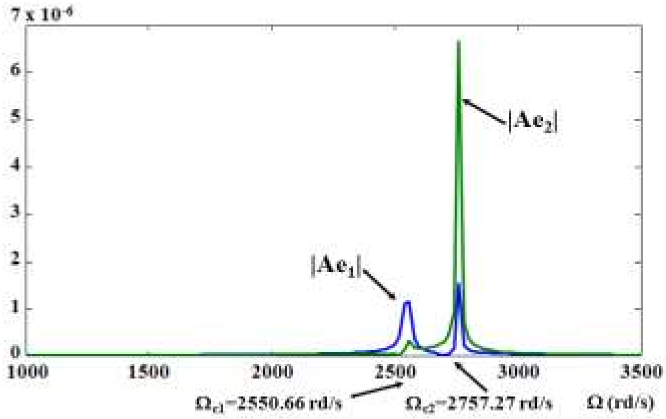

Figure 8(a) : Response to an asynchronous force: rotor with one disc free-simply supported case.

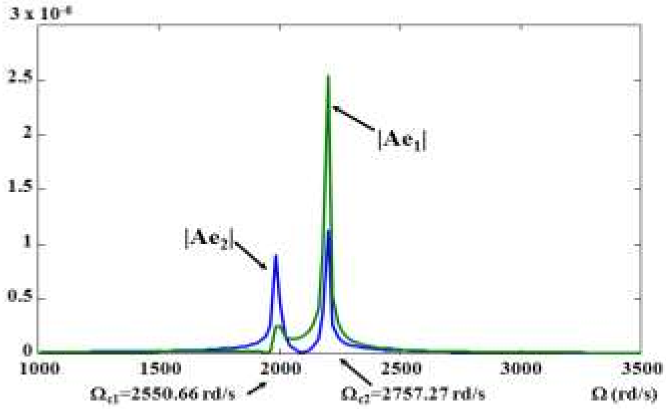

Figure 8(b) : Response to an asynchronous force: rotor with two discs free-simply supported case.

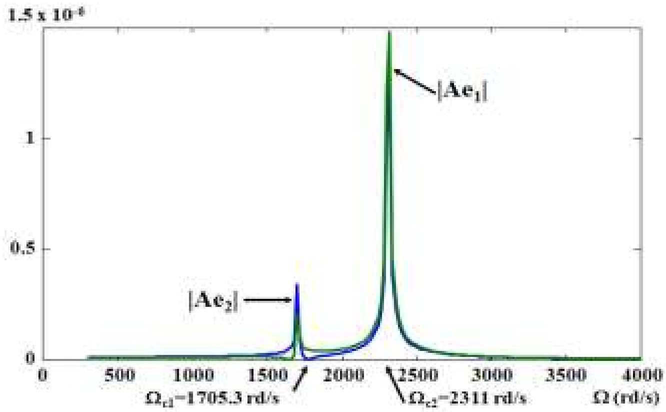

Figure 8(c) : Response to an asynchronous force: rotor with three discs free-simply supported case.

$$A_{e2} = \frac{(43{,}358.10^6 - 45{,}574\ \Omega^2)F}{453{,}636\ \Omega^4 - 26{,}479.10^8\Omega^2 + 2{,}255.10^{15}} \quad (78)$$

- Rotor with three discs:

$$A_{e1} = \frac{(52{,}015.10^6 - 57{,}31\Omega^2)F}{1158{,}86\Omega^4 - 36{,}9719.10^8\Omega^2 + 2{,}255.10^{15}},$$

$$A_{e2} = \frac{(43{,}358.10^6 - 57{,}31\Omega^2)F}{1158{,}86\Omega^4 - 36{,}9719.10^8\Omega^2 + 2{,}255.10^{15}} \quad (79)$$

In the free-simply supported case,

- Rotor with one disc:

$$A_{e1} = \frac{(99{,}4867.10^6 - 13{,}588\Omega^2)F}{173{,}4136\Omega^4 - 24{,}466.10^8\Omega^2 + 8{,}577.10^{15}},$$

$$A_{e2} = \frac{(86{,}216.10^6 - 13{,}588\Omega^2)F}{173{,}4136\Omega^4 - 24{,}466.10^8\Omega^2 + 8{,}577.10^{15}} \quad (80)$$

- Rotor with two discs:

$$A_{e1} = \frac{(99{,}4867.10^6 - 22{,}74\Omega^2)F}{450{,}934\Omega^4 - 39{,}5268.10^8\Omega^2 + 8{,}577.10^{15}},$$

$$A_{e2} = \frac{(86{,}216.10^6 - 22{,}74\Omega^2)F}{450{,}934\Omega^4 - 39{,}5268.10^8\Omega^2 + 8{,}577.10^{15}} \quad (81)$$

- Rotor with three discs:

$$A_{e1} = \frac{(99{,}4867.10^6 - 31{,}569\Omega^2)F}{552{,}261\Omega^4 - 45{,}555.10^8\Omega^2 + 8{,}577.10^{15}},$$

$$A_{e2} = \frac{(86{,}216.10^6 - 31{,}569\Omega^2)F}{552{,}261\Omega^4 - 45{,}555.10^8\Omega^2 + 8{,}577.10^{15}} \quad (82)$$

The amplitudes A_{e1} and A_{e2} for F=1N, are represented in figures (7, 8) where the critical speeds are the speeds that

12.

make the magnitude infinite and the denominator of a zero value.

CONCLUSION

In this work we have investigated the effect of the change in the boundary conditions on the vibration behavior of flexible rotor. The supported-free case is examined and compared with the supported-supported case of Lalanne and Ferrari.

On the one hand the results obtained by the Campbell diagram showed a net difference of critical speeds for these two cases.

On the other hand forces synchronous and asynchronous responses have given separate curves where you notice for the supported-free case of frequencies of overtones than those of the cases supported-supported.

It gives the possibility to work with greater speeds. Also it may be noted that playing on the boundaries conditions to allow avoiding the critical speeds for a given rotational speed and without changing the rotor structure.

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