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The use of shadow prices in capital rationing problem

Abstract

The aim of this article is to examine the use of linear programming to transfer funds from earlier to later periods by using the shadow prices as indicators in the transferring process, taking into consideration that each final budget period should not be greater than the sum of the starting budgets and the amount which can be transferred from its preceding periods. This method of transferring funds has two advantages, an increase in the objective function without using external resources. In some cases, the problem of accepting fractional projects can be overcome.

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ملخص

نحاول في هذا المقال توضيح كيفية استعمال البرمجة الخطية في ترشيد الموارد المالية وهذا باستعمال المعلومات المحصل عليها من الحلول الخاصة باختيار المشروعات الاستثمارية، وهذا باستغلال أسعار الظل (تكلفة الفرصة) وهذا بتحويل الموارد من السنوات السابقة للسنوات اللاحقة التي يكون فيها أسعار الظل فيها مرتفعة، وميزة هذه الطريقة هو رفع مستوى العائد في دالة الهدف دون استعمال موارد أخرى خارجية وكذلك من الممكن التغلب على قبول جزء من المشاريع بجعلها مقبولة كليا.

We can use the use of shadow prices (1) in linear programming techniques to increase the objective function when we change one of the constraints by adding some extra unit of resources while the other variables are held constant (2). This idea is only useful when there are some constraints in the problem which use the same resources. The use of shadow prices to improve the L.P. result in capital rationing case, one of the main reasons is that these shadow prices have no limits because we change more than one constraint at once. The second reason is that most of linear programming problems have different constraints. In capital rationing problem there is more than one budget constraint in the problem. The concept of transferring funds between periods was first suggested in the from of a model by professors Baumol (3) and Quandt in 1965. The firm should try to maximize the benefit of its available resources; this can be done by transferring funds from earlier to later periods. When the available budget is properly allocated then the firm can look for external resources to make the solution more optimal. The use of shadow prices in capital rationing is to transfer funds from earlier periods which have lower shadow prices to later periods if their shadow prices are higher than the preceding ones. If we have two periods in the capital rationing (4) problem and the first solution gives us these two values of shadow prices, D_1=1 for period 1, and D_2=2 for period 2. These two shadow prices indicate that the two periods have tight budgets. The most crucial budgeting period in the problem is period 2 where the opportunity of marginal Dinar is higher than the shadow price of period 1. In this kind of problem, if we transfer one unit of resources from period 1to period 2 we can increase our objective function without using more resources than the first budget. This, of course, assumes that the coefficients of the objective function and the constraints remain unchanged. I think the shadow prices could de used to transfer funds from earlier to later periods. This process should continue until all the period shadow prices are changed and there is no benefit resulting from a transfer of funds forward or backward. The theoretical solution is that all the shadows are equal at this point. In L.P.(5) however when an amount is transferred the shadow prices change with respect to the new opportunities which become feasible after the transfer takes place. In linear programming each component in the problem cam change these shadow prices when their values are changed. The discount rate has its effect on the objective function but we assume as mentioned previously that the project's cash flows gave been discounted at a known and unchanging vector of discount rate over the life of the projects; we also assume a certainty condition (6).

The question which we can now ask is: do the shadow prices have any relation when they are used in transferring funds from earlier to later periods? .when we transfer funds from one period to another, does our objective function increase by the difference between the two shadow prices or not? Alternatively, if we transfer funds from the higher to the lower shadow price, does our solution decrease by the difference between the two shadow prices?

These questions are examined in the next page. This method of transferring funds has one limitation in that, there are no limits in which the shadow prices are meaningful. This limitation makes it difficult for us to know when the shadow prices are going to be changed and at which point we should stop transferring funds from one period to another.

This method of transferring funds has two advantages:

1. An increase in the objective function without using external resources.
2. In some cases the problem of accepting fractional projects can be overcome.

The process should take the from of iterations until the value of shadow prices is changed and it becomes unbeneficial to transfer more funds. It is obvious that the process is unbeneficial when that shadow prices are lower than their precedents; since in such cases the transfer has the effect of reducing the objective function.

We demonstrate the use of prices in capital rationing case by using this examples which are chosen at random.

Example1: If we consider Lorie (7) and Savage example for two periods case:

Max $14X_1 + 17X_2 + 17X_3 + 15X_4 + 40X_5 + 12X_6 + 14X_7 + 10X_8 + 12X_9$

Subject to:

$12X_1+54X_2+6X_3+6X_4+30X_5+6X_6+48X_7+36X_8+18X_9 \leq 50.$

$3X_1+7X_2+6X_3+2X_4+35X_5+6X_6+4X_7+3X_8+3X_9 \leq 20$

$0 \leq X_j \leq 1 \qquad X_j=1\ldots\ldots\ldots\ldots\ldots\ldots9$

The solution is:

NPV = 70,2727DA

$X_1=X_3=X_4=X_9=1$, $X_6=0{,}97$, $X_5=0{,}045$, $X_2=X_7=X_8=0$

$SL_1=SL_2=0$

The shadow prices of the budget constraints are:

$D_1=0{,}1364$ $\qquad D_2=1{,}8636$

The dual values of the projects constraints are:

$U_1=6{,}7727$, $U_3=5$, $U_4=10{,}4545$, $U_9=3{,}9545$

$U_2=U_5=U_6=U_7=U_8=0$

The shadow price of period 2 is higher than the shadow price in period 1. It is beneficial to transfer some funds from period 1 to period 2 increase our objective function.

If we suppose that the values of shadow prices are not going to be changed when we increase b_2 by one unit of resources while at the same time reducing period 1 by one pound, we expect our objective function will increase by the difference of the two shadow prices 1,7272 (1,8636-0,1364). The difficulty of the shadow prices in fact before transferring one pound will change twice. The only thing which we should do is to transfer small amounts to know at what point the shadow prices will be change. When we try to transfer 0,16DA to period 2 the solution is:

NPV = DA 70,5431

$X_1=X_3=X_4=X_6=X_9$, $X_5=0{,}00021$, $X_7=0{,}0$, $X_2=X_8=0$

$SL_1=SL_2=0$

The budget shadow prices are:

$D_1=0{,}2115$ $\qquad D_2=0{,}9615$

The dual values of the projects constraints are:

$U_1=8{,}5769$, $U_3=9{,}9615$, $U_4=11{,}8077$, $U_6=4{,}9615$, $U_9=5{,}3077$

$U_2=U_5=U_7=U_8=0$

If 0,16DA is transferred to period 2 the NPV should increase by an amount equal to 0,16 ×1,7272 = 0,276352 (where, 1,7272 is the difference between the two shadow prices =1,8636- 0,1364=1,7272).

On transfer of L0,16 to period 2, the actual increase in the NPV= 70,5431-70,2727 =0,2704. This computed actual increase in the NPV is less than the expected increase in its value using the first shadow prices by 0,2764-0,2704=0,0060. This difference indicates that there was a change in the shadow prices during computation on the NPV by the computer.

The actual increase in the objective function is due to the two differences in the shadow prices, viz:

1st difference(1st trial) =1,8636-0,1364 = 1,7272

2nd difference(2nd trial) =0,9615-0,2115=0,75

Assuming X is the amount transferred which is affected by the first difference and y, the amount transferred which is affected by the second difference. The total amount transferred is equal to 0,16= X+y.

Than, it is therefore possible to determine the point up to which the first set of shadow prices are used and the starting point for the second set of shadow prices.

1,7272 X +0,75 Y = 0,2704 (1)

but

X +Y= 16 X=0,16-Y (2)

We substitute the value of X in (1)

0,2704 =(0,16 –Y) 1,7272 +0,75Y

Y=0,006099 X=0,153901

The amount 0,153601 is the upper limit where the first difference is useful. After this point there is the starting point of the use of the second difference.

In our second solution we still have in period 2 the shadow price is higher than period 1 dual value. We can continue transferring funds to period 2. If we reduce our first period budget to 48,14 the increase the second budget to 21,86, this should be done by transferring another amount equal to 1,7DA.

The solution is:

NPV= 71,3178 DA

$X_1 = X_3 = X_4 = X_6 = 1$, $X_5 = 0{,}0612$, $X_9 = 0{,}9057$, $X_2 = X_7 = X_8 = 0$

$SL_1 = SL_2 = 0$

The budget shadow prices are:

$D_1 = 0{,}5556$ $D_2 = 0{,}6667$

The dual values of the projects constraints are:

$U_1 = 5{,}3333$, $U_3 = 9{,}6667$, $U_4 = 10{,}3333$, $U_6 = 4{,}6667$

$U_2 = U_5 = U_7 = U_8 = U_9 = 0$

The increase in NPV from the last trial is 0,7747 (71,3178 –70,5431). This increase is the result of the two difference between the shadow prices which are: the first difference which is 0,75, this difference is applied for one part of 1,7, and the other part is related to the new difference. 1111(0,6667-0,5556).

The next trial is to transfer L2,14 to period 2, this makes period one budget equal to; 46 and period 2 equal to 24.

The solution is:

NPV = 71,5556DA

$X_1 = X_3 = X_4 = X_6 = 1$, $X_5 = 0,1444$, $X_9 = 0,6481$, $X_2 = X_7 = X_8 = 0$

$SL_1 = 2$ $SL_2 = 0$

The budget shadow prices are:

$D_1 = 0,5556$ $D_2 = 0,6667$

The dual values of the projects constraints are:

$U_1 = 5,3333$, $U_3 = 9,6667$, $U_4 = 10,3333$, $U_6 = 4,6667$

$U_2 = U_5 = U_7 = U_8 = U_9 = 0$

The change in the objective function is increased by the amount equal to 0,2378 (71,5556-71,3178). In this trail the shadow prices are not changed and the amount transferred should increase the NPV by the amount of 0,2378 (0,1111 × 2,14 = 0,2378). This amount is exactly equal to the calculated increase in the objective function. When we consider the last two trials we notice that the increase in NPV is caused by the increase of the proportion of X_5 and decrease of X_9 fraction.

When we increase period 2 to 28DA and decrease period one to 42DA, the solution is:

NPV = 72DA

$X_1 = X_3 = X_4 = X_6 = 1$, $X_5 = 0,3$, $X_9 = 0,1667$, $X_2 = X_7 = X_8 = 0$

$SL_1 = SL_2 = 0$

The budget shadow prices are:

$D_1 = 0,5556$ $D_2 = 0,6667$

The dual values of the project constraints are:

$U_1 = 5,3333$, $U_3 = 9,6667$, $U_4 = 10,3333$, $U_6 = 4,6667$

$U_2 = U_5 = U_7 = U_8 = U_9 = 0$

The amount transferred should increase the objective function by 0,4444 (0,1111× 4). The actual increase in the NPV is equal to 0,4444 (72 – 71,5556). This result is the same as the expected one. If we try to transfer another 2 DA to period 2, this makes period 1 budget equal 40 and period 2 equal to 30.
The solution is:

NPV = 71,5758DA

$X_3 = X_4 = X_6 = 1$, $X_1 = 0,8788$, $X_5 = 0,3818$, $X_2 = X_7 = X_8 = X_9 = 0$

The budget shadow prices are:

$D_1 = 1,1212$ $D_2 = 0,1818$

The dual values of the project constraints are:
$U_3 = 9{,}1818$, $U_4 = 7{,}9091$, $U_6 = 4{,}1818$
$U_1 = U_2 = U_5 = U_7 = U_8 = U_9 = 0$

This result is lower than the previous one. The change in the shadow prices in the opposite direction makes one part of the amount have a positive effect and the remaining part has a higher negative effect which decreases the objective function. The point which we should stop transferring funds is somewhere between 40 and 42. There are two ways which could be used to find this point. The first way is to try to start increasing period 2 by an amount smaller than 2 DA repeatedly until the point can be found. The second method is to use linear functions to find this point. If we assume that between 42DA and 40DA there is only one change in the shadow prices. In this range 40DA and 42DA we have one difference 0,1111 which increases the objective function, at the same time the second difference is a loss equal to –0,9394 (-1,1212 + 0,1818). The decease in the objective function is equal to –0,4242 (71,5758 –72). We assume that X is the amount which increases the objective function and Y is the amount which decreases it.

Our liner function are: $-0{,}4242 = 0{,}1111X - 0{,}9324\,Y$ (1)

$X + Y = 2$ $X = 2 - Y$ (2)

If we substitute (2) in (1):

$-0{,}4242 = (2 - Y)\,0{,}1111 - 0{,}9394Y$

$1{,}0505\,Y = 0{,}6464$, $Y = 0{,}6153$, $X = 1{,}3847$

Then the maximum amount which we can add to increase our objective function is 1,3847 where 0,1111 difference in the shadow prices is applicable. After this point the shadow prices change and there is no benefit to transfer funds. When we solve the problem again with $b_1 = 40{,}6154$ and $b_2 = 29{,}3846$.

The solution is:

NPV = 72,1538DA

$X_1 = X_3 = X_4 = X_6 = 1$, $X_5 = 0{,}3538$, $X_9 = 0{,}0000019$, $X_2 = X_7 = X_8 = 0$

$SL_1 = SL_2 = 0$

The shadow prices of the budget shadow prices are:

$D_1 = 0{,}5556$ $D_2 = 0{,}6667$

The dual values of the project constraints are:

$U_1 = 5{,}3333$, $U_3 = 9{,}6667$, $U_4 = 10{,}3333$, $U_6 = 4{,}6667$

$U_2 = U_5 = U_7 = U_8 = U_9 = 0$

When we increase the amount in period 2 by 0,0004 to check that the previous result is optimal, this makes the period1 budget equal to 40,615 and period 2 equal to 29,385.

The solution is:

NPV = 72,1535DA
$X_3 = X_4 = X_6 = 1$, $X_1 = 0{,}9999$, $X_5 = 0{,}3539$, $X_2 = X_7 = X_8 = X_9 = 0$
The shadow prices of the budget shadow prices are:
$D_1 = 1{,}1212$ $D_2 = 0{,}1818$
The dual values of the project constraints are:
$U_3 = 9{,}1818$, $U_4 = 7{,}9091$, $U_6 = 4{,}1818$
$U_1 = U_2 = U_5 = U_7 = U_8 = U_9 = 0$

The decrease in the objective function is by an amount equal to 0,0003. This amount could also be calculated from the last difference between the two shadow prices.

$(-1{,}1212 + 0{,}1818) \times 0{,}0004 = 0{,}000375$

this small difference is the result of the approximation by the computer and some decimals awe not included in the optimum point.

This example of the two period demonstrates the use of shadow prices in transferring funds is meaningful even when there are no limits of using the shadow prices. Our solution is improved from 70,2727DA to 72,1588DA. The best combination of the projects where the number of fractional projects is very small in the second trial. These small proportion can be dropped and the firm accepts only the complete projects.

Multi-period projects:

This last example was intended to be over simple with the number of periods limited to two to show how we can use the shadow prices. In this multi-period example the difficult point is the one from which we should reduce the budget and to which we should transfer these amount,

The example is:

$$\text{Max } 10X_1 + 8X_2 + 20X_3 + 15X_4 + 30X_5 + 40X_6$$

Subject is:

$$4X_1 + 6X_2 + 10X_3 + 5X_4 + 12X_5 + 6X_6 \leq 25$$

$$5X_1 + 4X_2 + 7X_3 + 15X_4 + 10X_5 + 20X_6 \leq 28$$

$$4X_1 + 3X_2 + 10X_3 + 6X_4 + 15X_5 + 10X_6 \leq 30$$

$$4X_1 + 5X_2 + 15X_3 + 10X_4 + 12X_5 + 20X_6 \leq 36$$

$$0 \leq X_j \leq 1 \quad X_j = 1,\ldots\ldots 6$$

In this example we are going to include only the budget shadow prices which we need in the problem and we forget the dual vual of the projects constraints.

The solution for this problem is:

NPV= 70,8602DA
X_1 = 0,4083 , X_3 =0,81 , X_6 =0,4964 ,X_5 =1 , X_2 =X_4 =0
SL_1 = SL_2 = SL_4 =0 SL_3 = 0,0143
The shadow prices of the budget constraints are:
D_1 =0,2151 , D_2 =1,3978 , D_3 =0 , D_4 = 0,5376

We observe from this solution that D_2 is higher than D_1 and D_4 is higher than D_1 and D_3. The question is from which period should we start transferring funds? The most reasonable way is to start where you can get the highest increase when you transfer funds. The starting point should be transferring funds from period 1 to period 2. In this trail, in fact, we transferred funds from period one to period two and four to check if the use of shadow prices is still meaningful when we transfer to more than one period once.

If we transfer 1,2DA to period 1 and 0,4DA to period 4, the differences in these shadow prices are:
(1,3978 – 0,2151) = 1,1827DA between period 2 and 1; the other difference is between period 4 and 1is equal to 0,3225(0,5376–0,2151). From these two differences we expect an increase in the objective function equal to 1,54824DA (1,1827 ×1,2 +0,3225 × 0,4).

The first trail when b_1 =23,4 . b_2 =29,2 . b_3 =3 and b_4 =36,4.
The solution is:

NPV= 72,4086DA
X_1 = 0,14448 , X_3 =0,6681 , X_6 =0,69 ,X_5 =1 , X_2 =X_4 =0
SL_1 = SL_2 = SL_4 =0 SL_3 = 0,8401
The shadow prices of the budget constraints are:
D_1 =0,2151 , D_2 =1,3978 , D_3 =0 , D_4 = 0,5376

The budget shadow prices are not changed and the increase in NPV is equal to 1,5484 (72,4086 – 70,8602). This increase is equal to the one calculated from the shadow prices. The small difference is he result of the approximations of the result by the computer.

This trial is not really appropriate because the most economical way is not transfer funds to the shadow prices in descending order of magnitude (8). The conclusion from this trail, however, is that the shadow prices relationship still exists even if we transfer to more than one period at once. In fact we can transfer the slack of period three (LS_3 =0,8401) to period 4 but since the difference between period 2 and 2 shadow prices 1,1827 (1,3978 – 0,2151) is higher than the difference between period 4 and 3 shadow prices 0,5376 (0,5376 – 0) the next trail is to transfer from period 1 to period 2.

When we reduce the period 1 budget to 21,8 and increase period 2 to 30,8, the other two budgets 3 and 4 remain as before.

The solution is:

NPV= 74,3011DA

X_1 = 0,0013 , X_3 =0,4502 , X_6 =0, 8821 ,X_5 =1 , X_2 =X_4 =0

SL_1 = SL_2 = SL_4 =0 SL_3 = 0,6717

The budget shadow prices are:

D_1 =0,2151 , D_2 =1,3978 , D_3 =0 , D_4 = 0,5376

The increase in the objective function is done by an amount equal to 1,89232 (1,6 × 1,1827). This amount can be calculated from the difference between the last two trails.

We notice in these trials that the increase in the objective function is the result of the increase of the proportion of project 6 and the decease of the fractions of X_1 and X_3.

When we increase period 2 budget to 31,85 and reduce period 1 budget to 20,75, the other variables remain constant.

The solution is:

NPV= 75,3671DA

X_3 = 0,2778 , X_5 =1 , X_6 =0,9953 , X_1 = X_2 =X_4 =0

SL_1 = SL_2 = 0 SL_3 =2,269 SL_4 = 0,3272

The budget shadow prices are:

D_1 =0,7595 , D_2 =1,7722 , D_3 = D_4 = 0

The increase in the objective function was the result of two differences in the shadow prices,

The 1^{st} difference = (1,3978 – 0,2151) = 1,1827

The 2^{nd} difference = (1,7722 – 0,7595) = 1,0127

In order to improve this result we should transfer funds to period 2 and 1 where the shadow prices are the highest. We can continue transferring the unused funds of period 3 and 4 and the available funds in these periods.

When the shadow prices of any of these periods becomes the highest, this period should have from its preceding one. These links may reduce the number of trials and funds may be transferred to more profitable unknown future opportunities. The only condition which we should take into account is that each period budget should not be greater than its initial budget plus the amount which can be transferred from the earliest periods. For example, period 3 budget should at maximum equal to period 3 first budget plus the amount which are transferred from period 21 and 2.

The stopping point is when the solution is optimal and any amount transferred forward or backward will decrease our NPV.

When we continue this process of transferring funds forward and backward optimal point is when b_1 =20,9412 , b_2 = 32,0588 , b_3 = 29,5882 and b_4 =36,4118.

The solution is:

NPV= 75,3671DA
$X_5 = X_6 = 1$, X_3 = 0,2941 , $X_1 = X_2 = X_4$ =0
$SL_1 = SL_2 = 0$ SL_3 =1,647 SL_4 = 0,00028
The budget shadow prices are:
D_1 =0,7595 , D_2 =1,7722 , $D_3 = D_4$ = 0

When we check if this is the optimal solution, the transfer of a very small amount (0,002) to period 2 where the shadow prices is higher from period one which has lower shadow prices.

When b_1 =20,941 , b_2 =32,059 , b_3 =29,5582 , b_4 =36,4118.

The solution is:

NPV= 75,3671DA
$X_5 = X_6 = 1$, X_3 = 0,294 , X_1 = 0,00014 , $X_2 = X_4$ =0
$SL_1 = SL_2 = 0$ SL_3 =1,6472 SL_4 = 0,00057
The budget shadow prices are:
D_1 =1,3636 , D_2 =0,9091 , $D_3 = D_4$ = 0

The solution gives as the same NPV but the main change is that period 1shadow prices becomes higher than period two shadow prices. Then, this is the optimal solution and any transfer of funds forward or backward decreases our NPV, projects which can be accepted completely are X_5 an X_6, the small fraction of project X_3 can be dropped.

CONCLUSION

These examples demonstrate that in capital rationing case the shadow prices could be used when we want to transfer funds from earlier to later periods, there is a relationship between the amount transferred, the period shadow prices and the changing the objective function. This relationship is the corner stone of the method. The shadow prices are applicable in difference types of problems; two or more period problems. They are also useful in cases of tight budgets or budgets with slacks.

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