SOME ASPECTS OF QUANTUM TUNNELING EFFECT IN DE SITTER-SCHWARCHILD SPACE-TIME

Submited On 12/01/2014 – Accepted on 14/07/2014

Abstract

Using the tunneling effect method near the black hole horizon and the WKB approximation for both fermionic and bosonic particles, some aspects are shown and discussed in a de Sitter-Schwarchild space-time type.

Keywords : *quantum tunneling, cosmology*.

B. CHOUDER

Laboratory Of Mathematical And Subatomic Physics, Physics Department, **Faculty** of Exact Sciences, Frères Mentouri **University** Constantine, Alegria.

I. INTRODUCTION

In this paper we study the tunneling behavior of spin -1/2 fermions across the event horizon of the Schwarzschild-de Sitter black hole where effects of quantum gravity are taken into account , and using the Hamilton-Jacobi method [1,2]. However, for the case of the bosons we try to find an exact formula for $\partial_r I(\mathrm{r})$ for an arbitrary non-Abelian Yang-Mills theory [3,4].

II. HAWKING RADIATION OF SCHWARCHILD-DE SITTER BLACK HOLE

The metric of any spherically symmetric solution in a **Schwarzschild** form is:

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2(d\theta^2 + sin^2\theta\, d\varphi^2$$

The vacuum Einstein equations give a *linear* equation for $f(r)$, which has as solutions:

$$f(r) = 1 - 2a/r$$

and

$$f(r) = 1 - br^2$$

The first is a zero stress energy solution describing a black hole in empty space time, the second (with b positive) describes de Sitter space with a stress-energy of a positive cosmological constant of magnitude 3b. Superposing the two solutions gives the de Sitter– Schwarzschild solution:

$$f(r) = 1 - 2a/r - br^2$$

We remind that the two parameters a and b give the black hole mass and the cosmological constant respectively.

III. FERMIONS

The fermion's motion is determined by the generalized Dirac equation ref. [1]

$$\begin{bmatrix} i\gamma^0\partial_0 + i\gamma^i\partial_i(1-\beta m^2) + i\gamma^i\beta\hbar^2(\partial_j\partial^j)\partial_i \\ +\frac{m}{\hbar}(1+\beta\hbar^2\partial_j\partial^j - \beta m^2) \\ +i\gamma^\mu\Omega_\mu(1+\beta\hbar^2\partial_j\partial^j - \beta m^2) \end{bmatrix}\psi = 0 \quad (1)$$

Where :

$$\Omega_\mu \equiv \frac{i}{2}\omega_\mu{}^{ab}\Sigma_{ab}$$
$$\Sigma_{ab} = \frac{i}{4}[\gamma^a,\gamma^b] \quad ,\{\gamma^a,\gamma^b\} = 2\eta^{ab} ,$$
$$\omega_\mu{}^a{}_b = e_\nu{}^a e^\lambda{}_b \Gamma^\nu_{\mu\lambda} - e^\lambda{}_b\partial_\mu e_\lambda{}^a$$

We assume that the wave function of the spin up state has the form:

$$\psi = \begin{pmatrix} A \\ 0 \\ B \\ 0 \end{pmatrix} exp\left(\frac{i}{\hbar}I(t,r,\theta,\emptyset)\right) \qquad (2)$$

and

$$e_\mu{}^a = diag(\sqrt{f}, 1/\sqrt{f}, r, r\sin\theta),$$

$$:\gamma^t = \frac{1}{\sqrt{f(r)}}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix},$$

$$\gamma^\theta = \sqrt{g^{\theta\theta}}\begin{pmatrix} 0 & \sigma^1 \\ \sigma^1 & 0 \end{pmatrix},$$

$$\gamma^r = \sqrt{f(r)}\begin{pmatrix} 0 & \sigma^3 \\ \sigma^3 & 0 \end{pmatrix},$$

$$\gamma^\emptyset = \sqrt{g^{\emptyset\emptyset}}\begin{pmatrix} 0 & \sigma^2 \\ \sigma^2 & 0 \end{pmatrix} \qquad (3)$$

With

$$\sqrt{g^{\theta\theta}} = \frac{1}{r}, \sqrt{g^{\emptyset\emptyset}} = \frac{1}{r\sin\theta},$$

σ^i are the Pauli matrices. Our purpose is to find the solutions of eqn (1). First substituting the wave function eqn (2) and the matrices eqn (3) into the generalized Dirac equation eqn (1). Since we are working With the WKB approximation we obtain 4 Hamilton-Jacobi equations:

$$\begin{aligned} &-iA\frac{1}{\sqrt{f}}\partial_t I - B(1-\beta m^2)\sqrt{f}\partial_r I \\ &-Am\beta\left[g^{rr}(\partial_r I)^2 + g^{\theta\theta}(\partial_\theta I)^2 + g^{\emptyset\emptyset}(\partial_\emptyset I)^2\right] \\ &+B\beta\sqrt{f}\partial_r I\left[g^{rr}(\partial_r I)^2 + g^{\theta\theta}(\partial_\theta I)^2 + g^{\emptyset\emptyset}(\partial_\emptyset I)^2\right] \\ &+Am(1-\beta m^2) = 0 \end{aligned} \quad (4)$$

$$\begin{aligned} &iB\frac{1}{\sqrt{f}}\partial_t I - A(1-\beta m^2)\sqrt{f}\partial_r I \\ &-Bm\beta\left[g^{rr}(\partial_r I)^2 + g^{\theta\theta}(\partial_\theta I)^2 + g^{\emptyset\emptyset}(\partial_\emptyset I)^2\right] \\ &+A\beta\sqrt{f}\partial_r I\left[g^{rr}(\partial_r I)^2 + g^{\theta\theta}(\partial_\theta I)^2 + g^{\emptyset\emptyset}(\partial_\emptyset I)^2\right] \\ &+Bm(1-\beta m^2) = 0 \end{aligned} \quad (5)$$

$$A\left\{\begin{array}{c} -(1-\beta m^2)\sqrt{g^{\theta\theta}}\partial_\theta I \\ +\beta\sqrt{g^{\theta\theta}}\partial_\theta I\left[g^{rr}(\partial_r I)^2 + g^{\theta\theta}(\partial_\theta I)^2 + g^{\emptyset\emptyset}(\partial_\emptyset I)^2\right] \\ -i(1-\beta m^2)\sqrt{g^{\emptyset\emptyset}}\partial_\emptyset I \\ +i\beta\sqrt{g^{\emptyset\emptyset}}\partial_\emptyset I\left[g^{rr}(\partial_r I)^2 + g^{\theta\theta}(\partial_\theta I)^2 + g^{\emptyset\emptyset}(\partial_\emptyset I)^2\right] \end{array}\right\} = 0 \quad (6)$$

$$B\left\{\begin{array}{c} -(1-\beta m^2)\sqrt{g^{\theta\theta}}\partial_\theta I \\ +\beta\sqrt{g^{\theta\theta}}\partial_\theta I\left[g^{rr}(\partial_r I)^2 + g^{\theta\theta}(\partial_\theta I)^2 + g^{\emptyset\emptyset}(\partial_\emptyset I)^2\right] \\ -i(1-\beta m^2)\sqrt{g^{\emptyset\emptyset}}\partial_\emptyset I \\ +i\beta\sqrt{g^{\emptyset\emptyset}}\partial_\emptyset I\left[g^{rr}(\partial_r I)^2 + g^{\theta\theta}(\partial_\theta I)^2 + g^{\emptyset\emptyset}(\partial_\emptyset I)^2\right] \end{array}\right\} = 0 \quad (7)$$

To find the relevant solution we perform the separation of variables as follows:

$$I = -\omega t + W(r) + \Theta(\theta, \emptyset)$$

eqn (6)and eqn (7)They are identical after divided respectively by A and B

$$\left(\sqrt{g^{\theta\theta}}\partial_\theta\Theta + i\sqrt{g^{\emptyset\emptyset}}\partial_\emptyset\Theta\right)$$

$$\left[\beta g^{rr}(\partial_r W)^2 + \beta g^{\theta\theta}(\partial_\theta\Theta)^2 + \beta g^{\emptyset\emptyset}(\partial_\emptyset\Theta)^2 - (1-\beta m^2)\right] = 0$$

$$\left(\sqrt{g^{\theta\theta}}\partial_\theta\Theta + i\sqrt{g^{\emptyset\emptyset}}\partial_\emptyset\Theta\right) = 0$$

eqn (4)and eqn (5) They are identical after cancelling A and B and give

$$A_6(\partial_r W)^6 + A_4(\partial_r W)^4 + A_2(\partial_r W)^2 + A_0 = 0$$

With

$$A_6 = \beta^2 f^4$$

$$A_4 = \beta f^3(m^2\beta + 2\beta Q - 2)$$

$$A_2 = f^2[(1-\beta m^2)^2 + \beta(2m^2 - 2m^4\beta - 2Q + \beta Q)]$$

$$A_0 = -m^2(1-\beta m^2 - \beta Q)^2 f - \omega^2$$

$$Q = g^{\theta\theta}(\partial_\theta\Theta)^2 + g^{\emptyset\emptyset}(\partial_\emptyset\Theta)^2 = 0$$

Neglecting the higher orders of β and solving the above equations at the event horizon yields

$$W(r) = \pm\int(\Gamma.\Lambda)\,dr$$

where

$$\Gamma = \frac{1}{f}\sqrt{m^2(1-2\beta m^2)f + \omega^2}$$

and

$$\Lambda = (1 + \beta\left(\left(m^2 + \frac{\omega^2}{f}\right)\right)$$

With:

$$f(r) = 1 - \frac{2a}{r} - br^2$$

The tunneling rate [5] of the spin -1/2 fermion crossing the horizon is

$$\Gamma = \frac{P_{(emission)}}{P_{(absorption)}} = \frac{\exp(-2\mathrm{Im}I_+)}{\exp(-2\mathrm{Im}I_-)} = \frac{\exp(-2\mathrm{Im}W_+ - 2\mathrm{Im}\Theta)}{\exp(-2\mathrm{Im}W_- - 2\mathrm{Im}\Theta)}$$

the Hawking temperature

$$T = \frac{\hbar}{4}(ImW(r))^{-1}$$

IV. FERMIONS

The equation of motion give by

$$0 = \nabla^\vartheta F^a_{\mu\vartheta} = g^{\vartheta\alpha}\left[\partial_\alpha F^a_{\mu\vartheta} - \Gamma^\lambda_{\alpha\mu}F^a_{\lambda\vartheta} - \Gamma^\lambda_{\alpha\vartheta}F^a_{\mu\lambda} + gf^{abc}A^b_\vartheta F^c_{\alpha\mu}\right]$$

Where we defined

$$F_{\mu\vartheta}^{a} = \partial_{\mu} A_{\vartheta}^{a} - \partial_{\vartheta} A_{\mu}^{a} + g f^{abc} A_{\mu}^{b} A_{\vartheta}^{c}$$

We assume the vector field

$$A_{\mu}^{a} = a_{\mu}^{a} e^{\frac{-i}{\hbar} I}$$

Before simplifying slightly, We obtained finally

$$0 = g^{tt} \left[\frac{1}{\hbar^2} a_t^a (\partial_t I)^2 \right]$$

$$+ g^{rr} \left[\frac{1}{\hbar^2} a_t^a \partial_r I \partial_r I + \Gamma_{rt}^{t} \partial_r a_t^a - \Gamma_{rt}^{t} g f^{abc} A_t^b A_r^c - g \frac{i}{\hbar} f^{abc} A_r^b a_t^c \partial_r I \right]$$

Taking the semi classical limit we obtain

$$(\partial_t I)^2 = f^2 (\partial_r I)^2 \Rightarrow \partial_t I = \pm f \partial_r I$$

We perform the separation of variables as follows:

$$I(r, t, \theta, \varphi) = \Omega t + W(r) + I'(\theta, \varphi)$$

we obtain

$$\Omega = \pm f \frac{\partial W(r)}{\partial r}$$

$$W(r) = \pm \Omega \int \frac{dr}{f}$$

The tunneling rate is given by

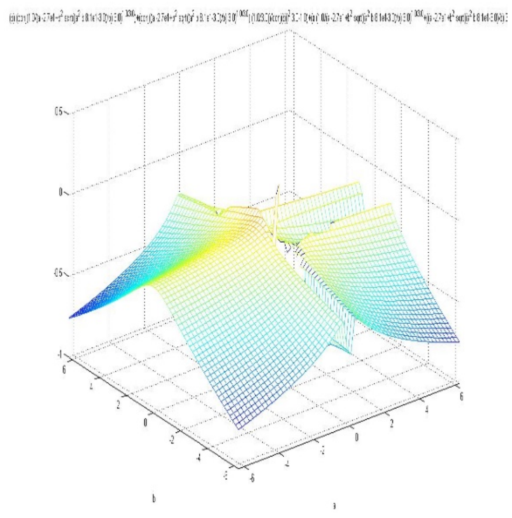
$$\Gamma = \frac{P_{(emission)}}{P_{(absorption)}} = \frac{\exp(-2\mathrm{Im} I_+)}{\exp(-2\mathrm{Im} I_-)}$$

$$= \frac{\exp(-2\mathrm{Im} W_+)}{\exp(-2\mathrm{Im} W_-)}$$

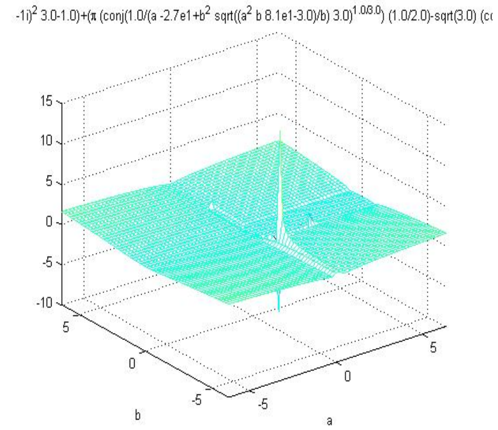
And therefore, the Hawking temperature reads:

$$T = \frac{\hbar}{4} \left(Im \int_0^r \frac{dr}{f} \right)^{-1}$$

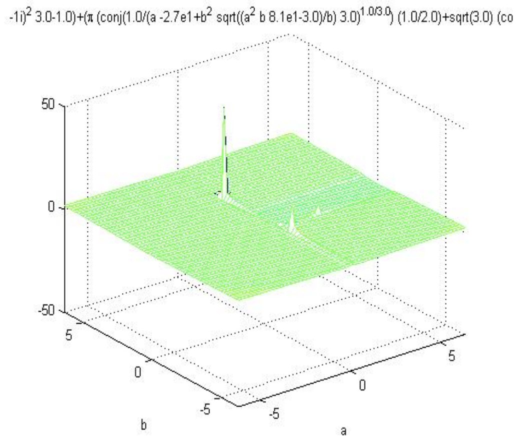
It is worth to mention that we have used Matlab[6] to solve this problem , the equation $r - 2a - br^3 = 0$ gives 3horizons r_1, r_2, r_2 for each we calculate the temperature of Hawking near r_1, r_2, r_2 . After we draw the corresponding curves $T_{1,2,3} = f(a, b)$ we get



The temperature near r_1



The Hawking temperature near r_2



The Hawking temperature near r_3

CONCLUSION

We have calculated the Hawking radiation temperature of spin 1/2 particles in the 4-dimensional Schwarzschild-de Sitter spacetime with Hamilton-Jacob method. As it is the case of the spin 1 particles.The tunneling rate and Hawking temperature were represented.

ACKNOWLEDGMENT

We are very grateful to the Algerian Ministry of education and research as well as the DGRSDT for the financial support.

REFERENCES

[1] Deyou Chen ,Houwen Wu and Haitang Yang, *Fermion's tunneling with effects of quantum gravity, (2013) [arXiv: 1305.7104v2 [gr-qc]]*

[2] K. Nozari and M. Karami, *Minimal length and generalized Dirac equation, Mod. Phys. Lett. A 20 (2005) 3095 [arXiv:0507028[hep-th]]*.

[3] B R Majhi,,S Samanta Ann. Phys.325:2410-2424 (2010) [arXiv:0901.2258]

[4] Alexandre Yale . *Exact Hawking Radiation of Scalars, Fermions, and Bosons Using the Tunneling Method Without Back-Reaction* ,(2011) *[arXiv: 1012.3165v3 [gr-qc]]*

[5] P. Mitra, *Hawking temperature from tunnelling formalism*, Phys. Lett. B 648 (2007) 240[arXiv:0611265[hep-th]].