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HAWKING TEMPERATURE NEAR LYRA BLACK HOLE'S HORIZONS

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Abstract

A model of the Lyra universe is presented. Using the tunneling effect approach, the Hawking radiation temperature near the black hole horizon is calculated.

Keywords: *Lyra geometry, black holes, Hawking radiation, Tunneling effect.*

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I- INTRODUCTION

Einstein developed [8] the general theory of relativity to unify gravity with other fundamental forces, but in this theory, gravitation is described in terms of Riemannian geometry, which could not only help to unify gravitation and electromagnetism in a single space-time geometry [1]. For that reason, Lyra [2] proposed in 1951 a modifications on Riemannian geometry (Lyra's geometry) by introducing a gauge or scale function which removes the non-instability condition of a vector under parallel transport. Soon after, Sen [3] and Sen with Dun in 1971[5] constructed an analog of the Einstein field equation based on Lyra's geometry as:

$$R_{ij} - \tfrac{1}{2}g_{ij}R + \tfrac{3}{2}\phi_i\phi_j - \tfrac{3}{2}g_{ij}\phi_k\phi^k = 8\pi G T_{ij} \qquad (1)$$

ϕ_i is the displacement field vector where:

$$\phi_i = (\beta, 0,0,0) \qquad (2)$$

β is a constant or a time-depending function. Further more, Sen and Dunn [3] gave a series type solutions to the static vacuum field equations. Retaining only a few terms in their solutions, we find that their solutions correspond to black holes (Lyra black holes).

In this paper, we study the Hawking radiation [4] of Lyra black hole. For that we proceed to analyze the Dirac equation in Lyra space-time and use the tunneling method because The Hawking's effect is a phase phenomenon. Such tunneling approach uses the fact that the WKB approximation of the tunneling probability for the classical forbidden trajectory from inside to outside the horizon is:

$$\Gamma \propto e^{-(2/\hbar)Im(I)} \qquad (3)$$

where I is the classical action of the trajectory, to leading order in $\hbar$.

II. LYRA BLACK HOLES

Let us consider a static spherically symmetric metric:

$$ds^2 = e^{\nu}dt^2 - e^{\lambda}dt^2 - r^2(d\theta^2 + sin^2\theta d\phi^2) \quad (4)$$

Sen and Dunn [1] are defined e^{ν} and e^{λ} to obtain solutions to the field equations (04) as:

$$\begin{cases} e^{\nu} = D + C\phi(r) \\ e^{\lambda} = \frac{Ar^4(\phi')^2}{D + C\phi(r)} \end{cases} \qquad (5)$$

Where

$$\phi = \textstyle\sum_{n=0}^{\infty} a_n r^{-n} \qquad (6)$$

A, B, C are arbitrary constant. The coefficients $a_n$are given by a_0. a_3 is arbitrary and $a_2 = 0$; a_n $(n \succ 0)$ are determined by:

$$0 = a_{n-1}[(D + Ca_0)(n-1)(n-4)] - Aa_1\sum(k-1)(n-k+1)a_{k-1}a_{n-k+1} - A\sum[(l-1)a_{l-1}]\,[\sum(k-1)(n-l-k+3)a_{k-1}a_{n-l-k+3}] - \sum(n-1)(2l-n-1)a_{n-l}a_{l-1} \qquad (7)$$

Retaining only a few terms, we have:

$$C^2 = \frac{2a_3}{a_1}, C + Da_0 = 1, a_1 = \pm\frac{1}{\sqrt{A}}, \frac{C}{\sqrt{A}} = M \qquad (8)$$

Where $M' = 2M$ is mass of black hole.
Thus, one can write e^{ν} and e^{λ} as:

$$\begin{cases} e^{\nu} = 1 - \frac{M}{r} + \frac{M\sqrt{A}a_3}{r^3} + \frac{M^2\sqrt{A}a_3}{r^4} \\ e^{\lambda} = \frac{\sigma^2}{1 - \frac{M}{r} + \frac{M\sqrt{A}a_3}{r^3} + \frac{M^2\sqrt{A}a_3}{r^4}} \end{cases} \qquad (9)$$

Where

$$\sigma^2 = 1 - \frac{6\sqrt{A}a_3}{r^2} - \frac{8M\sqrt{A}a_3}{r^3} + \frac{9Aa_3^2}{r^4} \qquad (10)$$

If we take the time component in (04), we can easily prove that the singularity locates at r=0, and the horizons correspond to $e^{\nu} = 0$; at that moment we get an equation in 4^{th} order,

$$r^4 - Mr^3 + pMr + pM^2 = 0 \qquad (11)$$

with $p = \sqrt{A}a_3$. This means that we will obtain four roots:

$$\begin{cases} r_1 = \frac{1}{2}\left[(a-l) + \sqrt{(a-l)^2 - 4(b+m)}\right] \\ r_2 = \frac{1}{2}\left[(a-l) - \sqrt{(a-l)^2 - 4(b+m)}\right] \\ r_3 = \frac{1}{2}\left[(a+l) + \sqrt{(a+l)^2 - 4(b-m)}\right] \\ r_4 = \frac{1}{2}\left[(a+l) - \sqrt{(a+l)^2 - 4(b-m)}\right] \end{cases} \quad (12)$$

Where

$$a = \frac{M}{2}, l = \frac{-(b+p)M}{2\sqrt{m}}, m = b^2 - pM^2,\ b = \frac{S}{12} + \frac{5pM^2}{S} \quad (13)$$

With

$$S = \left[108pM^2(p+M^2) + 12pM\sqrt{\left(-1338pM^2 + 81(M^4+p^2)\right)}\right]^{\frac{1}{3}} \quad (14)$$

b represents the real solution of the equation

$$b^3 - \frac{5}{4}pM^2 b - \frac{1}{8}(pM^2 + p^2M^2) = 0 \quad (15)$$

the two other solutions are given by the expressions

$$\begin{cases} b' = -\frac{S}{24} - \frac{5pM^2}{2S} + I\frac{\sqrt{3}}{4}\left(S - \frac{10pM^2}{S}\right) \\ b'' = -\frac{S}{24} - \frac{5pM^2}{2S} - I\frac{\sqrt{3}}{4}\left(S - \frac{10pM^2}{S}\right) \end{cases} \quad (16)$$

Only two roots between the solutions (12) are positive; see (Figure 1)

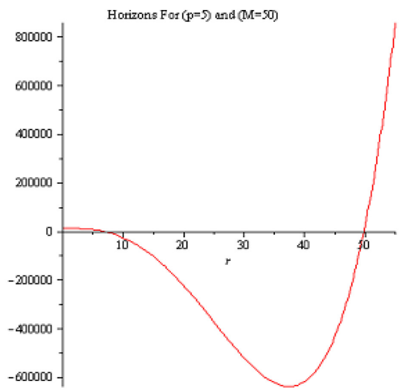

Figure 1: positive roots of e^{ν}

They correspond to the black hole's horizons:

$$\begin{cases} r_+ = r_1 = \Theta_1 + \Theta_2 \\ r_- = r_2 = \Theta_1 - \Theta_2 \end{cases} \quad (17)$$

Where

$$\Theta_1 = \frac{M}{4}\left(1 + \frac{B+p}{B^2 - pM^2}\right) \quad (18)$$

And

$$\Theta_2 = \frac{\sqrt{3}}{6}\sqrt{\frac{3}{4}M^2\left(\frac{B+p+1}{\sqrt{B^2-pM^2}}\right)^2 - \frac{60pM^2}{S} - 9(B^2+pM^2) - S} \quad (19)$$

where

$$B = \frac{S}{12} + \frac{5pM^2}{S} \quad (20)$$

Using Maple, r_+ *and* r_- are shown respectively in 3-dimensions with (Figure 2-a) and (Figure 2-b):

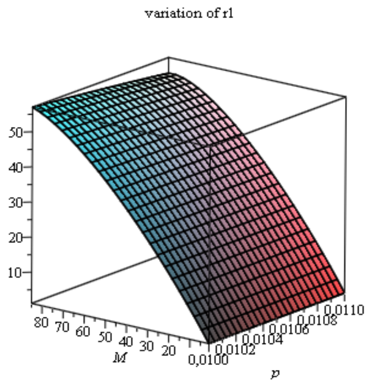

Figure 2-a: 3D-variation of r_+

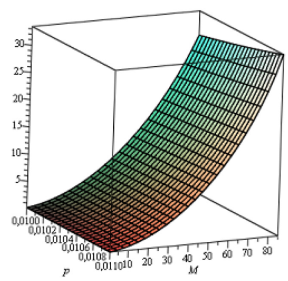

Figure 2-b: 3D- variation of r_-

We can also show the variation of the horizons r_+ and r_- with the curvature parameter of the space-time p and the mass of black hole M in two dimensions by (figures 3 -a, -b, -c and -d):

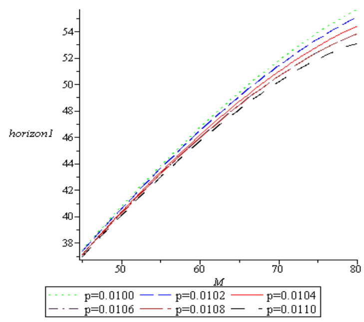

Figure 3-a : variation of r_+ with M in different values of p

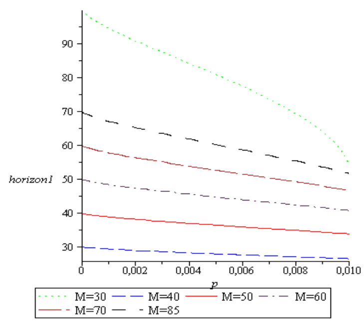

Figure 3-b : variation of r_+ with p in different values of M

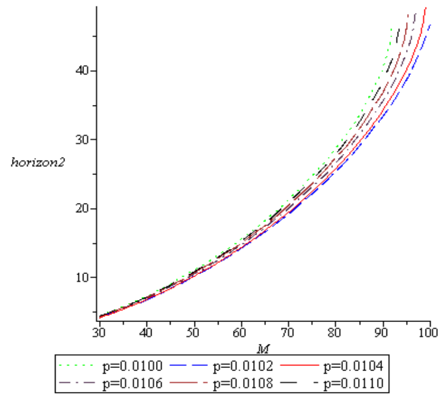

Figure 3-c : variation of r_{-} with p in different values of M

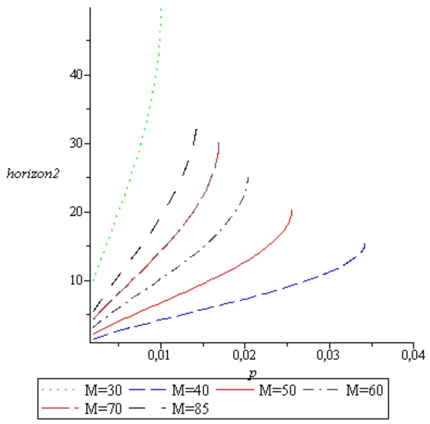

Figure 3-d : variation of r_{-} with M in different values of p

III. DIRAC EQUATION

Now we calculate the fermion's Hawking radiation from the apparent horizons of Lyra black holes via the tunneling formalism. For this we use the massless spinor field $\Psi(t,r,\theta,\varphi)$ obeyed the general covariant Dirac equation:

$$i\gamma^{\mu} D_{\mu}\Psi(t,r,\theta,\phi) = 0 \qquad (21)$$

where D_{μ} is the spinor covariant derivative is defined by

$$D_{\mu} = \partial_{\mu} + \frac{i}{2}\omega_{\mu}^{ab}\Sigma_{ab} \qquad (22)$$

and ω_{μ} is the spin connection, which can be given in terms of the tetrads e_{μ}^{a}.

The matrices $\gamma^{\mu} = \gamma^{a} e_{\mu}^{a}$ satisfy the Clifford algebra,

$$[\gamma_a, \gamma_b]_+ = 2\eta_{ab}\mathrm{I}_{4x4} \qquad (23)$$

and they are selected as

$$\gamma^0 = i\begin{pmatrix} I_{2x2} & 0 \\ 0 & -I_{2x2} \end{pmatrix}, \ \gamma^1 = \begin{pmatrix} 0 & \sigma^3 \\ \sigma^3 & 0 \end{pmatrix} \qquad (24)$$

$$\gamma^2 = \begin{pmatrix} 0 & \sigma^1 \\ \sigma^1 & 0 \end{pmatrix}, \qquad \gamma^3 = \begin{pmatrix} 0 & \sigma^2 \\ \sigma^2 & 0 \end{pmatrix} \qquad (25)$$

With

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad (26)$$

σ_i are the Pauli matrices satisfying the usual relation:

$$\sigma_i\sigma_j = I_{2x2}\delta_{ij} + i\epsilon_{ijk}\sigma_k \qquad (27)$$

$i,j,k = 1,2,3.$

In order to get the Dirac γ^{μ} matrices which are expressed in terms of the tetrads, we first define a tetrad of orthogonal vector e_{μ}^{a} where:

$$\eta_{ab} e_{\mu}^{a} e_{\nu}^{b} = g_{\mu\nu} \qquad (28)$$

Here $(a,b) \equiv (0,1,2,3)$ and $(\mu,\nu) \equiv (t,r,\theta,\varphi)$. The simplest choice of tetrads is given in the following matrix form:

$$e_{\mu}^{a} = \begin{pmatrix} e^{\nu/2} & 0 & 0 & 0 \\ 0 & e^{\nu/2} & 0 & 0 \\ 0 & r & 0 & 0 \\ 0 & 0 & rsin\theta & 0 \end{pmatrix} \qquad (29)$$

so

$$e_{a}^{\mu} = \begin{pmatrix} e^{-\nu/2} & 0 & 0 & 0 \\ 0 & e^{-\nu/2} & 0 & 0 \\ 0 & \frac{1}{r} & 0 & 0 \\ 0 & 0 & \frac{1}{rsin\theta} & 0 \end{pmatrix} \qquad (30)$$

With theses tetrads, it turns out that :

$$\begin{cases} \gamma^t = e^{-\nu/2}\gamma^0 \\ \gamma^r = e^{-\lambda/2}\gamma^1 \\ \gamma^{\theta} = \frac{1}{r}\gamma^2 \\ \gamma^{\phi} = \frac{1}{rsin\theta}\gamma^3 \end{cases} \qquad (31)$$

we can also write the matrix γ^5 in this way:

$$\gamma^5 \stackrel{\text{def}}{=} i\gamma^t\gamma^r\gamma^{\theta}\gamma^{\phi} = \frac{ie^{-(\nu+\lambda)/2}}{r^3 sin\theta}\gamma^0\gamma^1\gamma^2\gamma^3 \qquad (32)$$

III.A. HAWKING TEMPERATURE

To calculate the Hawking temperature, let us employ the following ansatz for the spin-up Dirac field:

$$\Psi_{\uparrow}(t,r,\theta,\phi) = \begin{pmatrix} \Gamma(t,r,\theta,\phi) \\ 0 \\ \Omega(t,r,\theta,\phi) \\ 0 \end{pmatrix} e^{\frac{i}{\hbar}I_{\uparrow}(t,r,\theta,\phi)} \qquad (33)$$

It should be noted that the spin-down case is just analogous. In order to apply the WKB approximation, we can plug the ansatz (33) into the general covariant Dirac equation (21) [4], it turns out that the term in square brackets is of order $O(\hbar)$. Thus we do not need to work out its pricise form, since in the $\hbar \to 0$ limit it vanishes. So the equation (21) becomes:

$$\hbar\partial\Psi_{\uparrow}(t,r,\theta,\phi) + o(\hbar) = 0 \qquad (34)$$

one can arrive at the expression:

$$e^{-\nu/2}\begin{pmatrix}\Gamma\partial_t I_\uparrow\\0\\-\Omega\partial_t I_\uparrow\\0\end{pmatrix}+e^{-\lambda/2}\begin{pmatrix}i\Omega\partial_r I_\uparrow\\0\\i\Gamma\partial_r I_\uparrow\\0\end{pmatrix}+\frac{1}{r}\begin{pmatrix}0\\i\Omega\partial_\theta I_\uparrow\\0\\i\Gamma\partial_\theta I_\uparrow\end{pmatrix}+\frac{1}{r\sin\theta}\begin{pmatrix}0\\-\Omega\partial_\phi I_\uparrow\\0\\-\Gamma\partial_\phi I_\uparrow\end{pmatrix}e^{\frac{i}{\hbar}I_\uparrow}=0 \quad (35)$$

Hence, we get the following equations system:

$$\begin{cases} t: -e^{-\nu/2}\Gamma\partial_t I_\uparrow + ie^{-\lambda/2}\Omega\partial_r I_\uparrow = 0\\ r: \frac{\Omega}{r}\left(i\partial_\theta I_\uparrow - \frac{1}{\sin\theta}\partial_\phi I_\uparrow\right)=0\\ \theta: -e^{-\nu/2}\Gamma\partial_t I_\uparrow + ie^{-\lambda/2}\Omega\partial_r I_\uparrow = 0\\ \phi: \frac{\Gamma}{r}\left(i\partial_\theta I_\uparrow - \frac{1}{\sin\theta}\partial_\phi I_\uparrow\right)=0 \end{cases} \quad (36)$$

Here Killing vector is time like $\chi = \partial_t$ is enough for this static black holes, it plays the role of Kodama vector for dynamical case.
To solve the above system, we use an other ansatz for the action $I_\uparrow$:

$$I_\uparrow = \int E dt + R(r) + J(\theta,\phi) + C \quad (37)$$

C is a constant.
This choice of $I_\uparrow$ leads to the following system of equations:

$$\begin{cases} t: -e^{-\nu/2}\Gamma E + ie^{-\lambda/2}\Omega R'(r) = 0\\ r: \frac{\Omega}{r}\left(iJ'_\theta(\theta,\phi) - \frac{1}{\sin\theta}J'_\phi(\theta,\phi)\right)=0\\ \theta: e^{-\nu/2}\Omega E + ie^{-\lambda/2}\Gamma R'(r) = 0\\ \phi: \frac{\Gamma}{r}\left(iJ'_\theta(\theta,\phi) - \frac{1}{\sin\theta}J'_\phi(\theta,\phi)\right)=0 \end{cases} \quad (38)$$

Where

$$\begin{cases} R'(r) = \frac{\partial R(r)}{\partial r}\\ J'_\theta(\theta,\phi) = \frac{\partial J(\theta,\phi)}{\partial\theta}\\ J'_\phi(\theta,\phi) = \frac{\partial J(\theta,\phi)}{\partial\phi} \end{cases} \quad (39)$$

For the second and the fourth equations in the system (38), we obtain the same results in the spin-down case [7], they imply that $J(\theta,\varphi)$ is complex function. However, as regards the first and third equations in (38), we can discuss two cases:

1. If $\Gamma = \pm i\Omega$, then we have:

$$\left(\mp e^{-\nu/2}E + e^{-\lambda/2}R'(r)\right)\Omega = 0 \quad (40)$$

which implies that:

$$R'(r) = \pm\frac{e^{-\nu/2}}{e^{-\lambda/2}}E \quad (41)$$

2. If $\Gamma = \pm\Omega$, then:

$$R'(r) = 0 \quad (41)$$

The case (41) corresponds to incoming particle absorbed in the classical limit with probability $\mathcal{P}_{incident} = 1$ [4-5], when the first case describes the emission process with the probability:

$$\Gamma \propto e^{-\frac{2}{\hbar}ImR(r)} \quad (42)$$

For that we need the imaginary part of the function $R(r)$, thus we have:

$$ImR(r) = \pm E Im\int\frac{e^{\lambda/2}}{e^{\nu/2}}dr = \pm E Im\int\frac{\sigma(r)}{e^{\nu}}dr \quad (43)$$

if we replace e^{ν} by its expression, we can write:

$$ImR(r) = \pm E Im\int\frac{r^4\sigma(r)}{(r-r_+)(r-r_-)H(r)}dr \quad (44)$$

where

$$H(r) = r^2 + (r_+ + r_- - M)r + \frac{pM^2}{r_+r_-}r + pM^2 \quad (45)$$

Using Residus theorem, we get :

$$R(r) = 2i\pi[Res(R(r),r_+) + Res(R(r),r_-)] \quad (46)$$

one can easily find $ImR(r)$ where we have two poles located at the horizons r_+ and r_- :

$$ImR(r) = \pm\frac{2\pi E}{r_+ - r_-}\left[\frac{r_+^4\sigma(r_+)}{H(r_+)} - \frac{r_-^4\sigma(r_-)}{H(r_-)}\right] \quad (47)$$

where

$$\begin{cases} \sigma(r_+) = \sqrt{1 - 6\frac{p}{r_+^2} - 8\frac{pM}{r_+^3} + 9\frac{p^2}{r_+^4}}\\ \sigma(r_-) = \sqrt{1 - 6\frac{p}{r_-^2} - 8\frac{pM}{r_-^3} + 9\frac{p^2}{r_-^4}} \end{cases} \quad (48)$$

Finally, taking the definition (42), we write:

$$\frac{E}{TH} = \pm\frac{2}{\hbar}ImR(r) \quad (49)$$

Here we can Distinguish two cases of the Hawking temperature: TH_{inner} caused only by r_- and TH_{outer} caused by both of the horizons r_+ and r_-:

$$TH_{inner} = \frac{\hbar}{4\pi}\left[\frac{r_-^2 - \frac{M}{2}}{r_-^3\sigma(r_-)}\right] \quad (50)$$

or:

$$TH_{inner} = \frac{\hbar}{4\pi\sigma(r_-)}\left[\frac{1}{r_-} - \frac{M}{r_-^3}\right] \quad (51)$$

and

$$TH_{outer} = \mp\frac{\hbar}{4\pi}\left[\frac{(r_+ - r_-)\left(r_+^2 - \frac{M}{2}\right)\left(r_-^2 - \frac{M}{2}\right)}{r_+^4\sigma(r_+)\left(r_-^2 - \frac{M}{2}\right) - r_-^4\sigma(r_-)\left(r_+^2 - \frac{M}{2}\right)}\right] \quad (52)$$

Using always the maple, we draw the variation of the Hawking temperature with the parameter of Lyra geometry and with the mass of the corresponding black hole presented in (Figure 4).

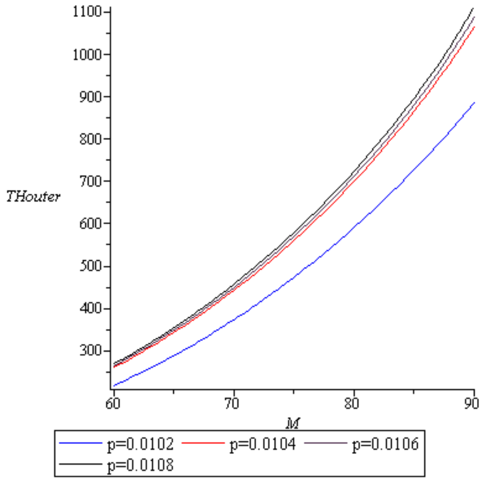

Figure 4 : Hawking radiation near Lyra black hole

III.B. PARTICULAR CASE

In this case, we consider one horizon for the Lyra black hole [1], i.e: the equation (11) has only one double positive root located at:

$$r_0 = \frac{12+\sqrt{p^2+204pM^2}}{6M} \quad (53)$$

for

$$72p^3 + 63p^2M + \sqrt{144p^2 + 204pM^2}(6p^2 + pM^2) = 9pM \quad (54)$$

in this case, the singularity is still at $r = 0$.
To calculate the Hawking temperature for the Lyra black holes with only one double positive horizon, the equation (43) becomes:

$$ImR(r) = \pm EIm \int \frac{r^4\sigma(r)}{(r-r_0)F(r)} dr \quad (55)$$

where

$$F(r) = r^2 + (2r_0 - M)r + \frac{pM^2}{r_0^2} \quad (56)$$

after integrating (55), we get:

$$R(r) = \pm 2\pi E \frac{d}{dr}\left(\frac{r^4\sigma(r)}{F(r)}\right)\Big|_{r=r_0} \quad (57)$$

this implies:

$$TH = \mp 2\pi \frac{r_0^3\sigma^2(r_0)[4+r_0(4r_0)-M]+2r_0\left(3p-\frac{6pM}{r_0}-\frac{9pM}{r_0^2}\right)}{\sigma(r_0)F^2(r_0)} \quad (58)$$

CONCLUSION

We conclude that the Hawking radiation near the black hole's apparent horizons depends of the space-time geometry (Lyra geometry) and the black hole's properties (mass).
With the Lyra geometry, we find the same results as the riemannien geometry ; the Hawking temperature increases by the increase of the mass of the black hole.

ACKNOWLEDGMENT

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