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EARTH'S TOMOGRAPHY WITH SUPERNOVA NEUTRINOS OSCILLATION IN THE LMA RANGE.

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Abstract

Low energy neutrinos can be used to probe the Earth's density from the study of the Earth's matter effects on their oscillation. In this work, we will show how this can be achieved with neutrinos coming from a future Galactic Supernova explosion ($D \sim 10\ Kpc$), using an analytic formula that describes the Earth's matter effects on their oscillation. We will focus, in this study, on the linear case where neutrinos travel short distances ($L \ll 1700\ km$) through the Earth, showing how a Tomogram of the Earth (a 2D image from a 3D body) can be created just by making use of the information obtained from the observation of the Earth's matter effects, which is the case for specific choices of the neutrino oscillation parameters and the neutrino mass schemes. In our work, we will treat the case where the neutrino oscillations are explained by the large mixing angle solution to the solar neutrino problem (LMA), combined with the normal mass hierarchy.

Keywords: *Supernova neutrinos, neutrino oscillation, Earth's matter effects.*

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INTRODUCTION :

Neutrino Tomography opens new possibilities to probe the Earth's interior, since neutrino physics is sensitive to the density of the matter travelled, unlike seismic geophysics which is sensitive to the density jumps. Depending on the propagation model, from the source to the detector, the neutrino tomography can be divided into two techniques [1], whether based on the interaction of neutrinos with the Earth's matter which manifests as the attenuation of the flux for high energy neutrinos (above TeV) [2], or on the Earth's matter effects on their oscillations for low energy ones (MeV to GeV). The first technique is known as NAT (Neutrino Absorption Tomography), and the second is known as NOT (Neutrino Oscillation Tomography).

Depending on the medium where they are first produced, NOT technique can be applied differently:

1. neutrinos produced in vacuum (Atmospheric neutrinos), or from man-made sources, do not have to go through dense matter, so, they reach the surface of the Earth as they are produced. The only matter that affects their oscillations is the Earth's ([1], [3], [4]). This method allows the construction of symmetric density profiles only.

2. neutrinos emerged from a stellar medium (The Sun or a Supernova), undergo the so-called MSW effect: they are produced at the core as flavor eigenstates, and have to pass –on their way out- by different layers (resonant layers) of different densities before travelling through the vacuum, to reach eventually the Earth as mass eigenstates.

In this work, we will focus on the second technique, which depends strongly on the Earth Matter Effects (EME) on the oscillation of Supernova (solar) neutrinos, trying to reproduce the Earth's density profile assuming that nothing is known about it, and that the only information we have is the neutrinos'.

An analytic formula that describes the EME on the oscillation of supernova neutrinos is obtained for the linear regime (, where a simple dependence of these effects on the matter induced potential allows one to notice that the EME analytic formula is a Fourier transform of , and then, by performing a Fourier transformation, we obtain the information sought for (. In doing so, some obstacles are met, most importantly, the lack of knowledge of the neutrino energies (especially from above). We will detail how it is dealt with, by invoking an iteration procedure that allows one to reach the "exact" profile after obtaining a series of potentials (four potentials for the solar case [5]).

In this paper, we will focus on Supernova neutrinos, which have higher energies from above (on the high energy tail of the spectrum). We will start (sec.II) with the neutrino evolution inside the SN, describing how the SN matter affects their oscillation (the MSW effect) before they leave it to travel through vacuum on their way to the Earth. After the neutrinos reach the Earth, their oscillation gets affected by the Earth's matter. We derive, in (sec.III) a theoretical formula that describes these effects, and we apply it to short distances travelled through the Earth (the linear regime). The results are presented in (sec IV) for three Earth density profiles: the Step-like, PREM (Preliminary Reference Earth Model) and an asymmetric step-like density profiles, showing by that how by making use of the Supernova neutrinos data, a tomogram of the Earth can be created.

II. Neutrino evolution inside the Supernova:

Neutrinos are produced in the core of the Supernova, and travel through its mantle and envelope on their way out. Being low energy neutrinos, they are transparent to the matter they go through (from the interaction perspective), thus one cannot expect much carried information, although, the oscillation phenomenon can bring information because the resonant oscillation depends on the density profile around the resonant point, which is the key ingredient to our study.

From the onset of the gravitational collapse to the explosion, neutrinos undergo several property changes, all happening before leaving the SN envelope. The original properties are expected from SN simulations, and any change whatsoever in them, testifies for neutrino conversion inside the SN and the SN matter effects on them.

Neutrino fluxes evolve with time. This main feature helps us to recognize the existence of three major phases in the evolution of the rates. Each phase is related to a well known process of emission [6].

The flavor eigenstate neutrino produced in the SN, emerges from it as mass eigenstate, after passing through a resonant region which is responsible for this kind of conversion.

In the case of solar neutrinos, only one resonant neutrino oscillation occurs in the star [7], however in the Supernova case, neutrinos go through two resonant points, because the core density is sufficiently high. The resonance density is proportional to the mass difference:

$$\rho_{res} \sim 1.4\times10^6 g/cc \left(\frac{\Delta m^2}{1eV}\right)\left(\frac{10\,MeV}{E}\right)\left(\frac{0.5}{Y_e}\right)\cos 2\theta \quad \text{(II. 1)}$$

- The layer at higher densities (H-resonance layer) which corresponds to Δm^2_{atm} is at

$$\rho_H \sim 10^3 - 10^4 g/cc \qquad \text{(II. 2)}$$

- The layer at lower densities (L-resonance layer), characterized by Δm^2_{sol} is at :

$$\rho_L = \begin{cases} 5-15\ g/cc \\ 10-30\ g/cc \\ < 10^{-4} g/cc \end{cases} \qquad \text{(II.3)}$$

Which correspond respectively to the LMA, SMA, and VO solutions to the solar neutrino problem.

The transition regions are far outside the core of the SN, and occur mainly in the outer layers of the mantle. The dynamics of conversions in the two resonance layers can be considered independently, and each transition is reduced to a two neutrino problem:

- In the H-resonance layer, the mixing U^m_{e2} associted with Δm^2_{sol} is suppressed by the matter. The suppression factor is:

$$\frac{U^m_{2e}}{U_{2e}} \sim \frac{\rho_L}{\rho_H} \lesssim 10^{-2} \qquad \text{(II.4)}$$

Correspondingly, the effects driven by Δm^2_{sol} are suppressed by a factor of two.

- In the L-resonance layer, the mixing associated with U^m_{3e} coincides with vacuum mixing ($U^m_{3e} \approx U_{3e}$).

Therefore, the level ν_3 does not participate in the dynamics: it decouples from the rest of the system, and the problem is reduced to a two state problem.

The dynamics of transitions in each layer is determined by the adiabaticity parameter γ:

$$\gamma = \frac{1}{2n}\left(\frac{\Delta m^2}{E}\right)^{1-\frac{1}{n}} \frac{\sin^2 2\theta}{(\cos 2\theta)^{1+\frac{1}{n}}} \left(\frac{2\sqrt{2}G_F Y_e}{m_N} A\right)^{1/n} \qquad \text{(II.5)}$$

Such that the probability that a neutrino jumps from one matter eigenstate to another (the flip probability) is given by [8]:

$$P_f = exp\left[-\left(\frac{E_{na}}{E}\right)^{2/3}\right] \qquad \text{(II.6)}$$

Where

$$E_{na} = \left(\frac{\pi}{12}\right)^{3/2} \frac{\Delta m^2 \sin^3 2\theta}{\cos^2 2\theta} \left(\frac{2\sqrt{2}G_F Y_e}{m_N} A\right)^{1/2} \qquad \text{(II.7)}$$

(Figure. 1) illustrates how we can divide the whole range of energy into three parts:

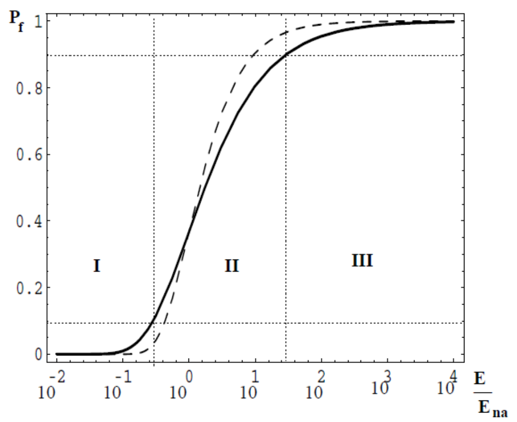

Fig. 1. The dependence of P_f on E/E_{na}(the solid line is for the density profile $\rho \sim r^{-3}$, and the dashed line is for a density profile $\rho \sim e^{-r}$) [8]

1. **Region I:** $(E/E_{na} < 10^{-1})$

Where pure adiabatic conversion occurs: $P_f \approx 0$.

2. **Region III:** $(E/E_{na} > 10^2)$

This region corresponds to a strong violation of adiabaticity: $P_f \approx 1$

3. **Region II:** $(E/E_{na} = 10^{-1} - 10^{2})$

In this region (the transition region), the adiabaticity breaking increases with E (P_f increses with the increase of the neutrino energy). For our case of density distribution profile ($\rho \sim r^{-3}$), the transition region does not depend strongly on the neutrino energy, compared to the case of the Sun ($\rho \sim e^{-r}$), where the transition region spans only about two orders of magnitude (P_f depends strongly on the neutrino energy).

The observable part of the SN neutrino spectrum spans about one order of magnitude. If the spectrum is in region I, completely adiabatic conversion occurs for the whole spectrum. In the region II, the conversion depends on the energy, however, the dependence is not strong over the relevant range of energies.

(Figure. 2) shows the contours of equal P_f in the $(\Delta m^2, \sin^2 2\theta)$-plot for energies E=5 MeV and E=50 MeV. The neutrinos belonging to the same line (determined by the couple $(\Delta m^2, \sin^2 2\theta)$) have the same P_f.

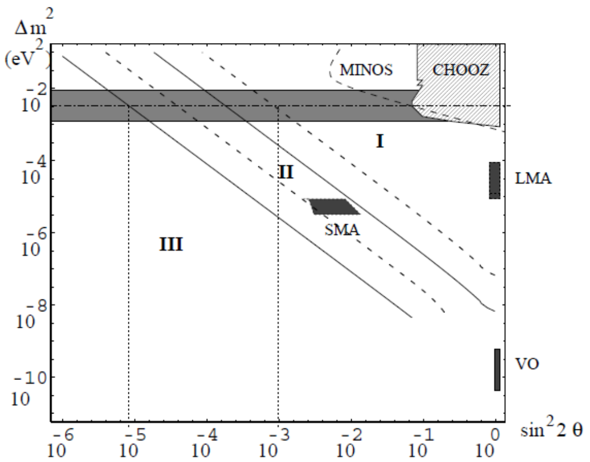

Fig. 2: Contours of equal P_f in the $(\Delta m^2, \sin^2 2\theta)$-plot for two different energies on the borders of the observable spectrum (Solid lines for 5 Mev, and dashed lines for 50 Mev). [8]

The contours of $P_f \approx 0.1$ (adiabatic transitions) and $P_f \approx 0.9$ (highly non-adiabatic transitions) divide the plot into three regions:

- The adiabatic region: is the region above the contour $P_f \approx 0.1$, where the adiabaticity is satisfied and strong flavor conversions occur. The LMA solution lies in this region.
- The transition region: is the region between $P_f \approx 0.1$ and $P_f \approx 0.9$ contours. In this region, the adiabaticity is partially broken, and the transitions are not complete. Moreover, the extent of transitions depends on the energy.
- The non-adiabatic region: is the region below the contour $P_f \approx 0.9$. The neutrino conversions are practically absent.

Each neutrino mass and flavor spectrum can be presented by two points in the $(\Delta m^2, \sin^2 2\theta)$-plot, which characterizes the layers H and L. One corresponding to $(\Delta m_{31}^2, \sin^2 2\theta_{e3})$, should lie in the atmospheric band, and the other one should lie in one of the "islands" corresponding to the solutions of the solar neutrino problem.

The H-resonance is in the adiabatic range for $\sin^2 2\theta_{e3} \approx 4|U_{e3}|^2 \gtrsim 10^{-3}$,

and in the transition region for

$10^{-5} \lesssim \sin^2 2\theta_{e3} \lesssim 10^{-3}$.

Now, to follow the neutrino conversion inside the SN, we need to reconstruct the neutrino mass spectrum (which is described by a two points in the $(\Delta m^2, \sin^2 2\theta)$-plane), but, due to the multiplicity of the possible schemes (six possible schemes of neutrino masses and mixings), which are related to the multiple solutions there are to the solar neutrino problem, the unidentified type of mass hierarchy and the unknown yet precise value of U_{e3}, we will limit this study to one possible scheme (of our choice) from which we extract the neutrino (antineutrino) survival probabilities.

We focus on the scheme of the LMA solution and the normal mass hierarchy, within which, the predictions depend on the value of U_{e3}.

The antineutrinos are represented on the same level crossing diagram, as neutrinos travelling through matter with "effectively" negative n_e, because their effective potential V has an opposite sign.

For the normal mass hierarchy, and as long as the solar neutrino solution is LMA, both resonances (L and H) lie in the neutrino channel (Figure. 3):

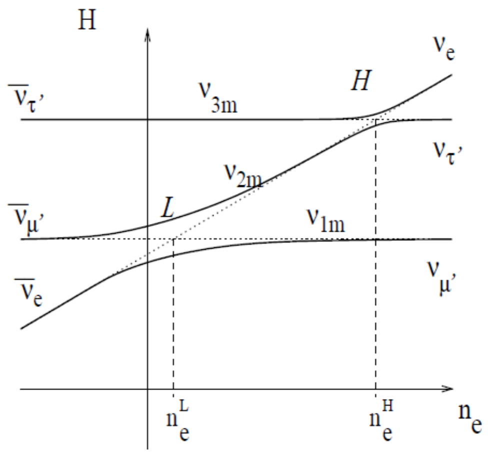

Fig. 3. Level crossing scheme for the LMA solution and the normal mass hierarchy [8].

Taking into account that the dynamics of transitions in the two resonance layers are independent, the fluxes of neutrino mass eigenstates at the surface of the star can be written down directly by tracing the path of the neutrino in the level crossing diagram. The fluxes are expressed in terms of P_L, P_H, $\bar{P}_L$, $\bar{P}_H$ (flip probabilities for neutrinos (antineutrinos) in L, H-resonance layers)

$$\begin{pmatrix} F_e \\ F_{\bar{e}} \\ 4F_x \end{pmatrix} = \begin{pmatrix} p & 0 & 1-p \\ 0 & \bar{p} & 1-\bar{p} \\ 1-p & 1-\bar{p} & 2+p+\bar{p} \end{pmatrix} \begin{pmatrix} F_e^0 \\ F_{\bar{e}}^0 \\ F_x^0 \end{pmatrix} \quad \text{(II.8)}$$

F_i^0 are the initial neutrino fluxes for the i-flavor as produced in the core of the star.

F_i are the neutrino fluxes at the surface of the star.

$p, \bar{p}$ are the survival probabilities of electron neutrinos (antineutrinos).

The neutrino fluxes arise from the central parts of the SN (high density regions) where all the mixings are highly suppressed. The initial neutrino flavor states coincide then with the matter eigenstates.

Using the level crossing scheme (fig. 3), we derive the following general expressions for the electron neutrinos survival probabilities [8]:

- For neutrinos:

$$p = |U_{e1}|^2 P_H P_L + |U_{e2}|^2 (P_H - P_H P_L) + |U_{e3}|^2 (1 - P_H) \quad \text{(II.9)}$$

- For antineutrinos

$$\bar{p} = |U_{e1}|^2 (1 - \bar{P}_L) + |U_{e2}|^2 \bar{P}_L \quad \text{(II.10)}$$

For the LMA solution, the solar neutrino data is explained via the $\nu_e \to \nu_2$ resonant conversion inside the sun with a large mixing angle:

$$sin^2 2\theta_{sol} = 0.7 - 0.95 \approx 1 \quad \text{(II.11)}$$

For the case of the LMA solution ($P_f \equiv P_L = \bar{P}_L \approx 0.1$) and the normal mass hierarchy (Fig.3), antineutrinos do not encounter any resonance ($\bar{P}_H = 0$), even though, there is a significant $\nu_e \leftrightarrow \nu_2$ conversion due to the large mixing angle! The evolution thus is adiabatic in both L and H layers in this channel.

The following transitions occur:

$$\bar{\nu}_e \to \bar{\nu}_1, \quad \bar{\nu}_{\dot{\mu}} \to \bar{\nu}_2, \quad \bar{\nu}_{\dot{\tau}} \to \bar{\nu}_3$$

The survival probability for $\bar{\nu}_e$ is then:

$$\bar{p} \approx |U_{e2}|^2 \approx cos^2\theta_{sol} \quad \text{(II.12)}$$

In the neutrino channel, $P_L \approx 0$ (LMA lies in region I)

$$p \approx |U_{e2}|^2 P_H + |U_{e3}|^2 (1 - P_H) \quad \text{(II. 13)}$$

The expression of the survival probability of ν_e contains thus the flip probability in the H-resonance layer. So, depending on the value of P_H, the neutrino survival probability takes values between

$$|U_{e3}|^2 \le p \le |U_{e2}|^2$$

Region I: ($P_H = 0$)

The level crossing scheme leads to the following transitions:

$$\nu_e \to \nu_3, \quad \nu_{\dot{\mu}} \to \nu_1, \quad \nu_{\dot{\tau}} \to \nu_2$$

From (II.9), the electron neutrino survival probability is then:

$$p \approx |U_{e3}|^2 \le 0.03 \quad \text{(II.14)}$$

The flavor transitions in this region are thus complete.

Region II:

In this region, the following transitions occur

$$\nu_e \to \nu_2, \nu_3 \quad \nu_{\dot{\mu}} \to \nu_1 \quad \nu_{\dot{\tau}} \to \nu_2, \nu_3$$

The neutrino survival probability is given by:

$$p \approx |U_{e2}|^2 P_H \quad \text{(II.15)}$$

Region III:

The H-resonance is inoperative ($P_H = 1$):

$$p \approx |U_{e2}|^2 \approx 0.2 - 0.4 \qquad \text{(II.16)}$$

The following transitions occur:

$$\nu_e \to \nu_2 \qquad \nu_\mu \to \nu_1 \qquad \nu_\tau \to \nu_3$$

Thus, for the normal mass hierarchy and the LMA solution, the survival probabilities for both neutrinos and antineutrinos are summarized in (Table. 1)

Regions where the value of P_H lies	p	$\bar{p}$
Region I	$\|U_{e3}\|^2$	$cos^2\theta_{sol}$
Region II	$P_H\, sin^2\theta_{sol}$	$cos^2\theta_{sol}$
Region III	$sin^2\theta_{sol}$	$cos^2\theta_{sol}$

Tab.1: Survival probabilities for the LMA solution ($P_L = \bar{P}_L = 0$)

In vacuum, and on the way to the Earth, any coherence between the mass eigenstates is lost due to the divergence of the wavepackets. Indeed, over a distance D, the two wavepackets corresponding to two mass eigenstates with a given Δm^2 and having an energy E separate from each other by a distance $\Delta d = \frac{\Delta m^2}{2E^2} D$. The lengths of the individual wavepackets are:

$\sigma \lesssim \frac{1}{T} \approx 10^{-11}\, cm$, Where T is the temperature of the production region.

So, even for the smallest $\Delta m^2 \sim 10^{-10}\, eV^2$, for $E = 10 MeV$, and $D = 10 Kpc$, the two wavepackets sepearte from each other by a distance $\Delta d = 10^{-2} cm$, which is way larger than the lengths of the individual wavepackets ($\Delta d \gg \sigma$). This leads to conclude that neutrinos are not affected by the vacuum they propagate through, and that they reach the surface of the Earth as incoherent fluxes of the mass eigenstate (mixed flavors).

III. Neutrino evolution inside the Earth :

The Earth Matter Effects on the oscillation of neutrinos have been extensively studied [9], whether making use of one detector [10], two or even more detectors [11].

The neutrino trajectory inside the Earth, before reaching the detector, depends on the location of the SN with respect to the detector on Earth.

We will imagine a scenario where the SN faces a detector, which measures the neutrino flux as they reach the surface of the Earth (the same that emerges from the SN envelope) before crossing the Earth (no Earth matter effects), and another one shielded by the Earth, which measures the flux with the Earth's matter effects (figure. 4).

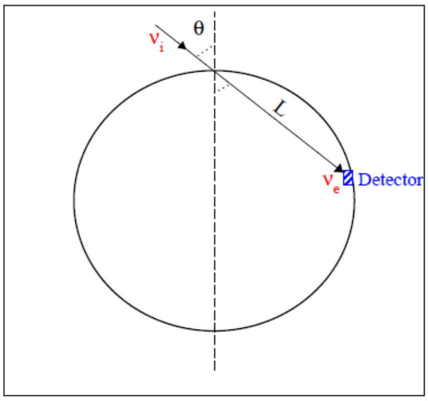

Fig. 4. Mass to flavor transition inside the Earth.

To every mass eigenstate flux reaching the Earth's surface, there are three contributions to this flux coming from the three falvor eigenstates produced in the SN.

The flux of ν_e at the detector (after traversing the Earth) is:

$$F_e^D = \sum_i P_{ie} F_i \qquad \text{(III.1)}$$

Where:

P_{ie} is the probability that a mass eigenstate ν_i entering the Earth reaches the detector as ν_e.

F_i: is the flux of mass eigenstates emerging from the SN envelope, and the same reaching the Earth's surface.

Equation (III.1) can be written also:

$$F_e^D = P^D F_e^0 + (1 - P^D) F_x^0 \qquad \text{(III.2)}$$

P^D is the neutrino survival probability at the detector.

$$P^D \equiv \sum_i a_i P_{ie} \qquad \text{(III.3)}$$

$$P \equiv \sum_i |U_{ei}|^2 a_i \qquad \text{(III.4)}$$

$$\begin{cases} a_1 = P_H P_L \\ a_2 = P_H(1-P_L) \\ a_3 = 1 - P_H \end{cases} \qquad \text{(III.5)}$$

From (III.2) the difference in the ν_e fluxes at the detector due to the propagation in Earth is:

$$F_e^D - F_e = (P^D - P)(F_e^0 - F_x^0) \qquad \text{(III.6)}$$

Where:

F_e^0 is the original neutrino flux (as produced inside the SN).

F_e the flux that reaches the Earth's surface.

F_e^D the electron neutrino flux after traversing the Earth.

$F_x^0 \equiv F_\mu^0 \equiv F_\tau^0$ are the original fluxes of the non electronic neutrinos in the SN.

P is the neutrino survival probability at the surface of the Earth (envelope of the Supernova).

From (III.3), (III.4) and (III.5):

$$P^D - P = P_H(P_{2e} - |U_{e2}|^2)(1-2P_L) + (P_{3e} - |U_{e3}|^2)(1 - P_H - P_H P_L) \qquad \text{(III.7)}$$

Inside the Earth ν_3 oscillates with a very small depth:

$$P_{3e} - |U_{e3}|^2 \lesssim \left(\frac{2E\, V_{Earth}}{\Delta m^2_{atm}}\right) sin^2 2\theta_{e3} \qquad \text{(III.8)}$$

V_{Earth} is the effective potential of ν_e in the Earth.

For SN neutrinos, $\frac{2E V_{Earth}}{\Delta m^2_{atm}} \lesssim 10^{-2}$.

Moreover, $sin^2 2\theta_{e3} \leq 0.1$. So that

$$P_{3e} - |U_{e3}|^2 \leq 10^{-3} \qquad \text{(III.9)}$$

So the second term in the equation (III.7) can be neglected, and the fluxes difference can be written for the normal mass hierarchy and the LMA solution case ($P_L = \bar{P}_L \approx 0$) :

- for neutrinos

$$F_e^D - F_e = P_H(P_{2e} - |U_{e2}|^2)(F_e^0 - F_x^0) \qquad \text{(III. 10)}$$

- for antineutrinos

$$F_{\bar{e}}^D - F_{\bar{e}} = (\bar{P}_{1e} - |U_{e1}|^2)(F_{\bar{e}}^0 - F_x^0) \qquad \text{(III. 11)}$$

P_{2e} is the probability that the mass eigenstate ν_2 entering the Earth, reaches the detector as ν_e.

Note that for the inverted mass hierarchy, the expressions of the flux differences have the same form, with substitution: $\nu \leftrightarrow \bar{\nu}$, $P_H \leftrightarrow \bar{P}_H$ and $P_L \leftrightarrow \bar{P}_L$.

Clearly the Earth matter effects are observed in the regions II and III (figure. 1) for the neutrino channel, where the transition in the H-resonance is purely non-adiabatic ($P_H \neq 0$), and for antineutrinos where the H-resonance is inoperative (for our chosen case).

The Earth matter effects are encoded in the quantity $(\bar{P}_{1e} - |U_{e1}|^2)$ for antineutrinos, and in $(P_{2e} - |U_{e2}|^2)$ for neutrinos. So we turn now to the calculation of this quantity.

- P_{2e} ($\bar{P}_{1e}$) is the probability that a mass eigenstate neutrino is found at the detector as an electron flavor neutrino after traversing a distance L inside the Earth.
- $|U_{e2}|^2$ ($|U_{e1}|^2$) is the projection of the mass eigenstate onto the electron flavor state.

To calculate the regeneration factor, we need to write the neutrino evolution equation in the flavor basis [12]

$$i\frac{d}{dt}\nu = [U\, diag(0, 2\delta, 2\Delta)\, U^+ + diag(V(t), 0, 0)]\nu \qquad \text{(III. 13)}$$

Where:

$\nu = (\nu_e, \nu_\mu, \nu_\tau)^T$ is the wave function of the neutrino system.

t: is the coordinate along the neutrino trajectory.

$V(t) = \sqrt{2} G_F N_e(t)$: is the neutrino matter induced potential.

For our purpose, it is convenient to go to the new basis defined through:

$\nu \equiv O_{23} \Gamma_\delta O_{13} \tilde{\nu}$. Then the neutrino evolution equation for the rotated state $\tilde{\nu}$ is:

$$i\frac{d}{dt}\begin{pmatrix}\tilde{\nu}_1\\ \tilde{\nu}_2\\ \tilde{\nu}_3\end{pmatrix}=\begin{pmatrix}2S_{12}^2\delta+C_{13}^2V(t) & 2S_{12}C_{12}\delta & S_{13}C_{13}V(t)\\ 2S_{12}C_{12}\delta & 2C_{12}^2\delta & 0\\ S_{13}C_{13}V(t) & 0 & 2\Delta+V(t)S_{13}^2\end{pmatrix}\begin{pmatrix}\tilde{\nu}_1\\ \tilde{\nu}_2\\ \tilde{\nu}_3\end{pmatrix}$$

Where:

$C_{ij}\equiv\cos\theta_{ij}$ and $S_{ij}\equiv\sin\theta_{ij}$

$\Delta=\frac{\Delta m_{32}^2}{4E}$

$\delta=\frac{\Delta m_{21}^2}{4E}$ (III. 14)

Neutrino evolution inside both the Supernova (Sun) and the Earth is adiabatic

($V\lesssim 2\delta\ll 2\Delta$) (III. 15)

In addition, since $S_{13}\ll 1$, we can to a very good accuracy neglect the (1-3) and (3-1) elements of the effective Hamiltonian compared to the (3-3) element:

$$i\frac{d}{dt}\begin{pmatrix}\tilde{\nu}_1\\ \tilde{\nu}_2\\ \tilde{\nu}_3\end{pmatrix}\approx\begin{pmatrix}2S_{12}^2\delta+C_{13}^2V(t) & 2S_{12}C_{12}\delta & 0\\ 2S_{12}C_{12}\delta & 2C_{12}^2\delta & 0\\ 0 & 0 & 2\Delta\end{pmatrix}\begin{pmatrix}\tilde{\nu}_1\\ \tilde{\nu}_2\\ \tilde{\nu}_3\end{pmatrix}$$ (III. 16)

This means that the third matter eigenstate decouples from the other two matter eigenstates. Besides, the (3-3) element in the Hamiltonian (III.16) shows that the Earth matter effects on the third mass eigenstate are negligible.

Let us now introduce the neutrino evolution matrix in the rotated basis according to:

$\tilde{\nu}(t)=\tilde{S}(t,t_0)\tilde{\nu}(t_0)$ (III.17)

$$\tilde{S}(t_0,t_0)=\begin{pmatrix}1 & & \\ & 1 & \\ & & 1\end{pmatrix}$$

The matrix $\tilde{S}(t,t_0)$ satisfies the same evolution equation as the rotated state $\tilde{\nu}$:

$$i\frac{d}{dt}\left(\tilde{S}(t,t_0)\tilde{\nu}(t_0)\right)=H\tilde{S}(t,t_0)\tilde{\nu}(t_0)$$

The decoupling of the third matter eigenstate implies that the evolution matrix is written as:

$$\tilde{S}(t,t_0)=\begin{pmatrix}\tilde{\alpha}(t,t_0) & \tilde{\beta}(t,t_0) & 0\\ -\tilde{\beta}^*(t,t_0) & \tilde{\alpha}^*(t,t_0) & 0\\ 0 & 0 & f(t,t_0)\end{pmatrix}$$ (III.18)

Where

$f(t,t_0)=exp[-2i\Delta(t-t_0)]$ (III.19)

Probabilities P_{1e} and P_{2e} in terms of $\tilde{\alpha}$, $\tilde{\beta}$:

$$\bar{P}_{1e}=C_{13}^2\left|C_{12}\,\tilde{\alpha}-S_{12}\tilde{\beta}\right|^2$$

$$P_{2e}=C_{13}^2\left|S_{12}\,\tilde{\alpha}+C_{12}\tilde{\beta}\right|^2$$

Finally, the Earth regeneration factor is expressed by:

$$P_{2e}-|U_{e2}|^2=C_{13}^2\left[\cos 2\theta_{12}\left|\tilde{\beta}\right|^2+\sin 2\theta_{12}Re\left(\tilde{\alpha}^*\tilde{\beta}\right)\right]$$ (III.20)

The expression (III.20), is valid for an arbitrary density profile, and reproduces all the analytic expressions obtained under simplified assumptions about the Earth's density profile (matter of constant density, three layers of constant densities and the adiabatic approximation) [12].

Note:

The quantities P_{ei}, $|U_{ei}|^2$ and P_{ie} satisfy the condition:

$$\sum_{i=1}^3 P_{ei}=\sum_{i=1}^3|U_{ei}|^2=\sum_{i=1}^3 P_{ie}=1$$ (III.21)

Since $P_{3e}=|U_{e3}|^2$ (Earth matter effects on the third mass eigenstate are negligible), (III. 21) gives:

$P_{1e}-|U_{e1}|^2=-(P_{2e}-|U_{e2}|^2)$ (III.22)

Now, to obtain the explicit expression for the regeneration factor, we have to find the explicit expressions for both $\tilde{\alpha}$, $\tilde{\beta}$ witch are valid for an arbitrary Earth density profile. The basic point is that the neutrino potential inside the Earth is small, and so can be considered as perturbation.

For that, we need to perform the perturbation theory in V [13] for the evolution matrix (III. 18), which requires -for its validity- the smallness of two dimensionless parameters:

- $\frac{V}{2\delta}$ which is indeed very small in our case (III. 15)

- $VL = \int_0^L V dx$ which can be large enough for long distances travelled by neutrinos through the Earth.

The effective Hamiltonian (III.16), can be decomposed as $\tilde{H}(t) = \tilde{H}_0 + \tilde{H}_1(t)$, Where $\tilde{H}_0$ and $\tilde{H}_1(t)$ are of zeroth and first order in V(t) respectively.

To first order in V(t), the evolution matrix $\tilde{S}(t, t_0)$ can be written as:

$$\tilde{S}(t,t_0) \approx \tilde{S}_0(t,t_0) - i\tilde{S}_0(t,t_0)\int_{t_0}^{t}\left[\tilde{S}_0(\acute{t},t_0)^{-1}\tilde{H}_1(\acute{t})\tilde{S}_0(\acute{t},t_0)\right]d\acute{t} \qquad \text{(III.23)}$$

For the zeroth-order matrix $\tilde{S}_0(t, t_0)$ we find ($\tilde{\alpha}_0$, $\tilde{\beta}_0$), we substitute them in (III.23) to find ($\tilde{\alpha}$, $\tilde{\beta}$), which we substitute in (III.20) to get the explicit expression of the regeneration factor [12]:

$$P_{2e} - |U_{e2}|^2 = \frac{c_{13}^4}{2}\sin^2 2\theta_{12}\int_0^L dx\ V(x)\sin[2\delta(L-x)] \qquad \text{(III. 24).}$$

x is the coordinate of the neutrino path length inside the Earth, and $L = 2R\cos\theta_Z$, where R is the Earth's radius and θ_Z is the zenith angle of the neutrino trajectory (Figure. 4).

We should remind here that the expression (III. 24) is valid for only short distances travelled by neutrinos through the Earth.

A more accurate formula can be obtained by replacing the integrant in (III.24), the in-vacuum oscillation phase, by the corresponding adiabatic one, i.e.

$$P_{2e} - |U_{e2}|^2 = \frac{c_{13}^4}{2}\sin^2 2\theta_{12}\int_0^L dx\ V(x)\sin\left[2\int_x^L w(\acute{x})d\acute{x}\right] \qquad \text{(III. 25)}$$

This result is obtained by performing the perturbation theory in $\frac{V}{2\delta}$ rather than in V. This theory requires only the smallness of $\frac{V}{2\delta}$, regardless of the neutrino path lengths inside the Earth (see [12] appendix A).

Equation (III. 24) can be written:

$$P_{2e} - |U_{e2}|^2 = \frac{c_{13}^4}{2}\sin^2 2\theta_{12} f(\delta) \qquad \text{(III. 26)}$$

Where

$$f(\delta) = \int_0^L dx\ V(x)\sin[2\delta(L-x)] \qquad \text{(III. 27)}$$

$f(\delta)$ has a Fourier integral form and actually means that in the limit of small V, this function is just the Fourier transform of the matter induced potential:

$$V(x) = \frac{4}{\pi}\int_0^\infty d\delta\ f(\delta)\sin[2\delta(L-x)] \qquad \text{(III. 28)}$$

There are some limitations of this result that should me mentioned:

- The found result is valid only in the limit $\frac{V}{2\delta} \ll 1$ and $VL \ll 1$.
- The function $f(\delta)$ has to be known precisely in the whole interval $0 \le \delta < \infty$ (which means (II.14) the whole energy interval $0 \le E < \infty$). Although, it is only measured in a finite energy interval, and with some experimental errors, because the detectors have finite energy resolution, and can give limited information on the energy of the incoming neutrinos, and the neutrino parameters.
- The precision in the neutrino oscillation parameters $\Delta m_{21}^2, \theta_{12}$ and θ_{13} are only known with certain experimental uncertainties.

IV. Procedure and results

For the matter density in the upper mantle of the Earth, $\rho \approx 3g/cm^3$, the condition $VL \ll 1$ leads to the upper limit on the allowed neutrino path lengths inside the Earth $L \ll 1700\ Km$. This condition will be relaxed in the study of the non linear regime.

Before getting started with the procedure, let us resume and rewrite the needed expressions for neutrinos and antineutrinos. We should remind here that we have limited the study to the LMA solution and the normal mass hierarchy.

- for neutrinos

From (III.26), (III.10) can be written

$$F_e^D - F_e = P_H\left(\frac{c_{13}^4}{2}\sin^2 2\theta_{12} f(\delta)\right)(F_e^0 - F_x^0) \qquad \text{(IV.1)}$$

- for antineutrinos

From (III.22) and (III.26), (III.11) can be written

$$F_{\bar{e}}^D - F_{\bar{e}} = \left(\frac{c_{13}^4}{2}\sin^2 2\theta_{12} f(\delta)\right)(F_{\bar{e}}^0 - F_x^0) \qquad \text{(IV.2)}$$

$f(\delta)$ is a quantity that is measured experimentally is only known for a finite interval $\delta_{min} \le \delta \le \delta_{max}$ and thus (III.14) a finite interval of neutrino energies: $E_{min} \le E \le E_{max}$.

The integral (III.28) requires that f is precisely measured for the whole interval $0 \le \delta < \infty$ (in the infinite interval of neutrino energies $0 \le E < \infty$), but since the neutrino energy is limited from above, we will see how this obstacle can be overcome.

First, let us consider an integral of the form (III.28)

$$\frac{4}{\pi}\int_{\delta_{min}}^{\delta_{max}} d\delta\, f(\delta)\sin[2\delta(L-x)]$$

This integral yields

$$\frac{1}{\pi}\int_0^L dy V(y)\left\{\frac{\sin[2\delta(x-y)]}{x-y} - \frac{\sin[2\delta(2L-x-y)]}{(2L-x-y)}\right\}\Bigg|_{\delta_{min}}^{\delta_{max}} \tag{IV. 3}$$

In order that this integral approaches the integral over infinite interval $(0 \le \delta < \infty)$:

$$\begin{cases} V \ll \dfrac{1}{L} \\ V \ll 2\delta \begin{cases} \delta_{max} \to \infty & \to \delta_{max} \gg \dfrac{1}{L} \gg V \\ \delta_{min} \to 0 & \to V \ll \delta_{min} \ll \dfrac{1}{L} \end{cases} \end{cases}$$

So, in the ideal case, we would like to have

$$\begin{cases} \delta_{min} L \ll 1 \\ \delta_{max} L \gg 1 \end{cases}$$

As we shall see, having large enough δ_{max} (E_{min}) does not pose any problem. However, in most situations of practical interest, the second condition is not satisfied ($\delta_{min} \gtrsim \frac{1}{L}$). We shall see that this difficulty can be readily overcome.

The fact of having a finite δ_{max} does not affect the procedure. It has been shown [5] that finite δ_{max} leads to a finite coordinate resolution of the reconstructed potential $V(x)$, as well as to small oscillations of the reconstructed potential around the true one. So, for good enough resolution, we can put $\delta_{max} \to \infty$ in our analytic formulas.

There are, though, two reasons why having a sufficiently small δ_{min} may be a fundamental problem:

1. Small δ_{min} implies large neutrino energies, and there are upper limits to the available neutrino energies.

2. The second obstacle is of more fundamental nature. The condition for which our main result is valid, i. e. $\frac{V}{2\delta} \ll 1$, may break down for too small δ_{min} (too high E_{max}). This gives a lower limit to values of δ_{min} one can use ($\delta_{limit} \le \delta_{min} \to E_{max} \le E_{limit}$).

To cure the problem posed by having a non zero δ_{min}, we put $\delta_{max} \to \infty$, then from the integral (IV.3) we find:

$$V(x) = \frac{4}{\pi}\int_{\delta_{min}}^{\infty} d\delta\, f(\delta)\sin 2\delta(L-x) + \frac{1}{\pi}\int_0^L dy V(y)\, F(x,y;2\delta_{min}) \tag{IV. 4}$$

Where the function $F(x,y;2\delta_{min})$ is defined as:

$$F(x,y;2\delta_{min}) = \frac{\sin 2\delta_{min}(x-y)}{x-y} - \frac{\sin 2\delta_{min}(2L-x-y)}{2L-x-y} \tag{IV. 5}$$

By comparing the integral (IV.4) with the integral (III.28), we can consider the second integral in (IV.4) as a compensating term for an error introduced in (III. 28) by having a non-zero lower limit in the integral over δ. This compensating integral cannot be calculated directly, since it contains the unknown potential $V(x)$. This problem can be cured by invoking a simple iteration procedure.

We first note that in the limit $\delta_{min} \to 0$, the first integral in (IV.4) yields $V(x)$ while the second one disappears. Therefore, for not too large values of δ_{min} (not too small values of E_{max}), the first integral gives a very good approximation to $V(x)$. One can use thus this value to obtain the result to the second part of (IV.4) (the compensating integral) to obtain the next approximation to $V(x)$:

$$V_0(x) = \frac{4}{\pi}\int_{\delta_{min}}^{\infty} d\delta\, f(\delta)\sin[2\delta(L-x)]$$

(IV.6)

We put:

$$I_0(x) = \frac{1}{\pi}\int_0^L dy V_0(y)\, F(x,y;2\delta_{min}) \tag{IV.7}$$

And then, we calculate the addition:

$$V_1(x) = V_0(x) + I_0(x) \tag{IV.8}$$

We follow the steps:

$$\begin{cases} I_{n-1}(x) = \frac{1}{\pi}\int_0^L dy V_{n-1}(y) F(x,y;2\delta_{min}) \\ V_n(x) = V_0(x) + I_{n-1}(x) \end{cases} \tag{IV. 9}$$

So, this yields a series of potentials $V_0(x), V_1(x), \dots, V_n(x)$, which for small enough δ_{min} converge to $V(x)$.

For this procedure to work though -in addition to the conditions posed for the linear regime- δ_{min} has to be chosen taking into account the following:

- First, it should be small enough (though possible to reach 0).
- Second, it should be smaller than a critical value δ_c, above which this iteration procedure fails: it yields potentials which, instead of approaching the true profile, they deviate from it ($\delta_{min} \le \delta_c \to E_c \le E_{max}$).

To proceed forward, we should remind that the quantity f (which differs from one profile to another) should be know from the experiment through (IV. 1), but since it is not available yet, here is what we do:

1. We choose a specific density profile of the Earth, from which we generate $f(\delta)$, by making use of the equation (III.27).
2. Then, we pretend that nothing is known about the Earth's density profile, and that the only thing we know is the function $f(\delta)$ (deduced by the step 1), which we use it as our input in the calculation to establish $V(x)$, making use of the formulas {(IV.6)-(IV.9)}.

The calculations are performed following [5], i. e. taking the same conditions used for the solar neutrinos, the only difference we apply here is in the neutrino energies, and more precisely E_{max} (δ_{min}).

The energy of SN neutrinos can be extended up to $70\ MeV$, for which the conditions to be satisfied are:

$$\begin{cases} \delta_{min} = 0.41\ L^{-1} \\ \delta_{max} = 300\ L^{-1} \end{cases} \tag{IV.10}$$

The critical value for δ_{min} was chosen to be $\delta_c \approx 2.4\ L^{-1}$ ($E_c = 12.5\ MeV$). i.e. the maximum energy has to be at least $12.5\ MeV$ for this iteration procedure to work, which is the case with SN neutrinos.

The precision was taken to be $\varepsilon \approx 3\times10^{-6}$ i.e. $|V(x) - V_n(x)| < \varepsilon$.

The study is performed on three different density profiles (different f) and are presented in what follows:

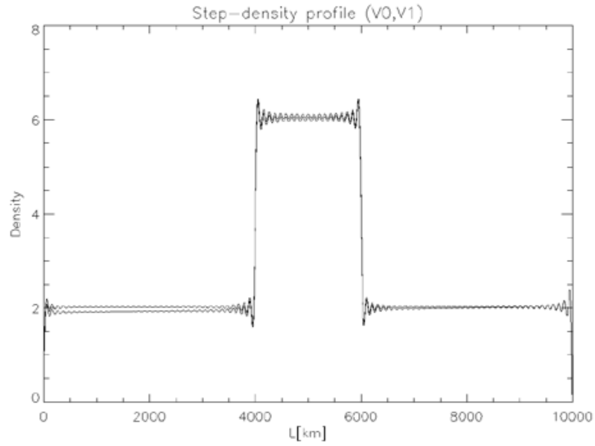

Fig. 5. Step density Profile: V_0, V_1($\delta_{min} = 0.41L^{-1}, \delta_{max} = 300L^{-1}$)

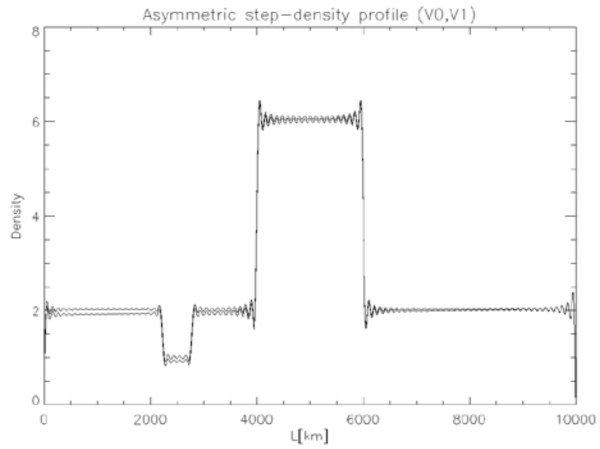

Fig. 6. Asymmetric step density Profile: V_0, V_1($\delta_{min} = 0.41L^{-1}, \delta_{max} = 300L^{-1}$)

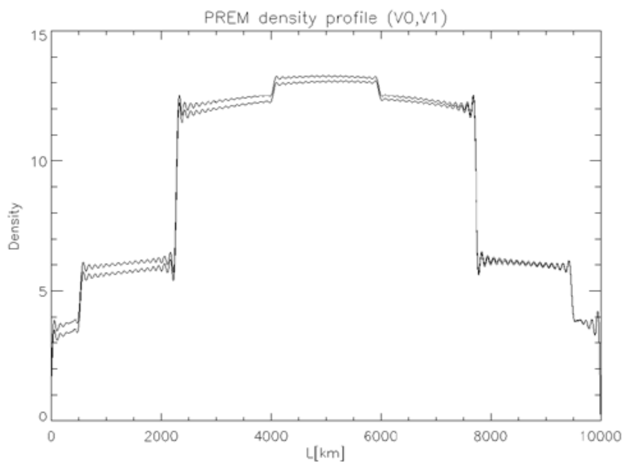

Fig. 7. PREM density Profile [14]: $V_0, V_1(\delta_{min} = 0.41L^{-1}, \delta_{max} = 300L^{-1})$

V. Analysis:

From the three profiles, the same observation is made:

- All of the Three profiles are produced "exactly" by the first iteration (V_1), regardless their shapes or their density distributions.
- The iterative potentials approach the true profile from below, and never exceed it.
- Even V_0 reproduces the same positions and magnitudes of the density jumps as the true one.
- The deviations of $V_n(x)$ from the exact potential are small at $x \approx L$ and largest at $x = 0$.

This approach not only reproduces the exact positions of the density jumps (already established from seismic geophysics data), but also gives the same magnitudes (the exact value of the densities) in the different layers, so it is complementary to seismic geophysics.

Even if the "true" profile is found from the 1st iteration, V_0 already gives the value of the density near the detector ($x \approx L$).

This procedure allows the reconstruction of even an asymmetric density profile, which can only be achieved for solar and supernova neutrinos due to their particularity of reaching the Earth as mass eignstates.

We should clarify here that by symmetric density profiles we mean profiles that have the same densities around the midpoint of the neutrino trajectory inside the Earth (L). They give rise to potentials that have the same property: $V(L-x) = V(x)$. This symmetry is only approximate; it is violated by inhomogeneities of the Earth's density distribution on short length scales ([15], [16]).

In the case of solar neutrinos [5], the exact profile was established after the fourth iteration (taking the same conditions here, except for the value of δ_{min} ($E_{max} = 30\ MeV \implies \delta_{min} = 3.33\times10^{-3}\ m^{-1}$), so clearly Supernova neutrinos give better (faster) information than Solar neutrinos. In other words, the higher the neutrino energy is, the faster the convergence to the true profile is, and the shorter the time this procedure takes.

VI. The non linear regime:

The previous study was based on the formula (III.24) of the regeneration factor (EME) which was derived after performing the perturbation theory in V. This theory requires the smallness of VL. i.e. short neutrino paths inside the Earth.

The more accurate formula (III.25), however, which was derived by the perturbation theory in $\frac{V}{2\delta}$, does not pose any condition on the length travelled, so, it can be used to probe the Earth's density over "realistic" distances. In other words, one has to employ the inversion procedure based on the expression:

$$f(\delta) = \int_0^L dx\, V(x) \sin\left[2\int_x^L w(\acute{x})d\acute{x}\right] \qquad \text{(VI.1)}$$

Where:

$$w(\acute{x}) = \sqrt{[cos2\theta_{12}\delta - V(x)/2]^2 + \delta^2 sin^2 2\theta_{12}}$$

Since (VI.1) was obtained without making any restriction on the distance travelled, ($\delta_{min}L \ll 1$ does not have to be satisfied here), the matter density profile cannot be found by invoking the previous iteration procedure. The problem becomes very difficult to solve.

The equation (VI.1) is a non-linear Fredholm integral equation of the first kind, and equations of this type are very difficult to solve [17], and need a dedicated study, since they belong to the class of "ill-posed" problems: their solutions are very unstable, and to arrive at a reliable result, one has to invoke special regularization procedures [18]. For non linear integral equations of the first kind, no universal regularization techniques exist.

The NOT technique can also be performed through another approach, which does not rely on the flavor to mass oscillation property that happens to Supernova neutrinos inside the SN, and which distinguishes Supernova (solar) neutrinos from other low energy ones. This approach not only can reveal information about the Earth's matter travelled, but moreover, it allows the probe of the entire Earth ($L = 2R_{Earth} = 12742\ km$) without putting boundaries on the distances neutrinos travel inside. It had been tackled in several papers ([3], [4] and [19]). It makes use of the direct problem rather than trying to solve the inverse one. It consists of generating random density distribution, dividing it into several layers and then comparing the obtained P_{ab} (theoretically obtained survival probability of the neutrino in the interval [a,b]) with simulated data for the "true" profile (figure. 8). Even though it looks more promising, this approach turned out to be not only a time consuming procedure, but also with limited accuracy!

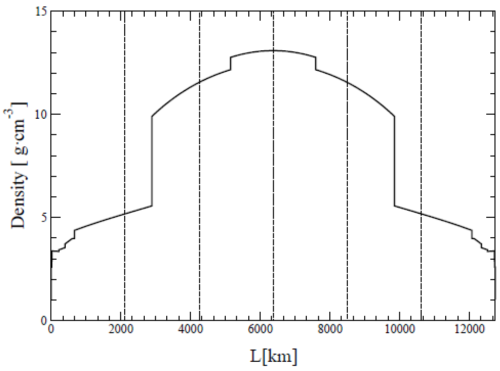

Fig. 8. a. PREM density Profile (step I)

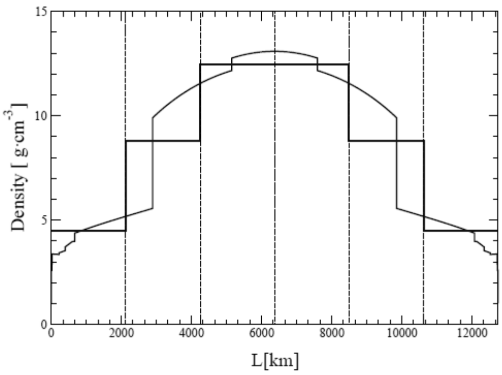

Fig. 8. b. PREM density Profile (step II)

VII. Conclusion :

In an attempt to resolve the inverse problem of neutrino oscillation, we have shown that neutrino oscillation in matter can serve as a powerful tool, not only in revealing information about the matter travelled, but also in probing the 'exact' density values at the different layers travelled. In the case of Supernova neutrinos, and based on their oscillation pattern inside the Supernova, namely the MSW effect, it is achieved through an analytic formula that describes the EME on their oscillation, which, for short distances travelled through the Earth (the linear regime), has a simple (direct) dependence on the neutrino matter induced potential (V). This latter was obtained after invoking an iteration procedure that helped us overcome the problem posed by the limitation (from above) of the neutrino energies. We managed thus, to establish the "true" density profile, for short distances ($L \ll 1700\ km$), in a faster (time conserving) way than the one achieved in the solar neutrino case.

We should mention here that for the chosen neutrino parameters (LMA solution to the solar neutrino problem for the normal mass hierarchy) the study was applicable because the EME on these neutrinos (and antineutrinos) are observed ((III.10),(III.11)), but for other neutrino mass scheme/neutrino oscillation parameters combinations (ex: inverted mass hierarchy/SMA), the situation is quite different: there are cases where these effects are not observed, thus, this study can no longer be applicable. It can, however, still be used the other way around: it can help us reduce the number of the possible neutrino mass schemes, and constrain the neutrino oscillation parameters, depending on the observation of the Earth matter effects.

To go further in this study, we can use the found results (the value of the density) to find the electron density number $N_e(x) = \frac{Y_e}{m_N}\rho(L)\ \ (Y_e \approx \frac{1}{2})$ which differs from one element to another, and thus obtain furthermore information about the Earth's composition. We can therefore whether consolidate or refute J. M. Herndon's controversial hypothesis about the existing of radioactive elements, namely Uranium and Thorium at the very center of the Earth's inner core, making thus birth to a sub-inner core that contains these lithophile elements, which –from a geochemical perspective- can never exist in the core of the Earth.

Herndon assumed that these elements are the cause of the Earth's magnetic field, and its instability, and that the energy realized from the decay of these radioactive elements, is the secret behind the heat generated by the Earth ([20], [21], [22]).

Even though this study is performed on distances way smaller than the Earth's distances we want to reach and explore, it allows us to go deeper than the depths reached (so far) by other techniques (digging). It can, thus, be used to reveal information about other "objects" that have "big" diameters and explore their deeper structures, but for the Earth and other objects that have distances comparable to the Earth's, the resolution of the non linear regime will be of a bigger benefit.

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