ESTIMATION FOR BOUNDED SOLUTIONS OF INTEGRAL INEQUALITIES SOME NEW NON-LINEAR RETARDER INTEGRO-DIFFERENTIAL INEQUALITIES.

Submited on 20/12/2018 – Accepted on 10/07/2019

Abstract

In this paper, we establish some new non-linear retarded integro-differential inequalities in tow and n independent variables.

Keywords: boundary Value Problems ; Retarded integro-differential Equations ; Partial integro-differential equations ; integral inequalities.

M.E SMAKDJI
H. KHELLAF

Department of Mathematics, Frères Mentouri University,Constantine, Algeria.

INTRODUCTION

The study of integral inequalities involving functions of one or more independent variables Is an important tool in the study of existence, uniqueness, bounds, stability, invariant manifolds and other qualitative properties of solutions of differential equations and integral equations (see : [1-6, 8,12]).

The study of integro-differential inequalities for functions of two or n variables is very significant and plays a role in the study of the existence and uniqueness of the solutions of Wendroff type integro-differential inequalities and equations as well as the boundedness of the solutions of the initial value problem of non-linear retarded hyperbolic partial integro-differential equations for functions of two or n variables [9-11].

Pachpatte [7] presented some new non-linear integro-differential inequalities of the Wendroff type for two-variable functions.

Lemma 1. (See Theorem 1 [7]) Let $\Phi(x,y)$ and $c(x,y)$ be non-negative continuous functions defined for $x\geq 0, y\geq 0$, and $\Phi(0,y)=\Phi(x,0)=0$ for which the inequality

$$\Phi_{xy}(x,y)\leq a(x)+b(y)+\int_0^x\int_0^y c(s,t)\left(\Phi(s,t)+\Phi_{xy}(s,t)\right)dsdt,$$

holds for $x\geq 0, y\geq 0$, where $a(x), b(y)\geq 0$; $a'(x), b'(y)\geq 0$ are continuous functions defined for $x\geq 0, y\geq 0$. Then

$$\Phi_{xy}(x,y)\leq a(x)+b(y)+\int_0^x\int_0^y c(s,t)\left[\frac{[a(0)+b(t)][a(s)+b(0)]}{[a(0)+b(0)]}\right]\exp\left(\int_0^s\int_0^t[1+c(\tau,\sigma)]d\tau d\sigma\right)dsdt$$

Our main aim here, motivated by the works of Pachppate [7] , Zhang, H. and Meng [12] , is to stablish some new non-linear retarded integro-differential inequalities for functions with tow and n independent variables which are useful in the analysis of certain classes of partial differential equations and integro-differential inequalities. Some applications of our results are also given

Throughout this paper, we denote $\mathbb{R}_+^n=[0,\infty[$ which is a subset of $\mathbb{R}_+^n$, $(n\geq 1)$. All the functions which appear in the inequalities are assumed to be real valued of n -variables $(n\geq 1)$which are non-negative and continuous. All integrals are assumed to exist on their domains of definitions.

We note $D=D_1D_2\cdots D_n$, whereD_i, for $i=1,2,\cdots,n$.

II. MAIN RESULTS

In this section, we present some results of non-linear retarded integro-differential inequalities in two independent variables.

Theorem 2. Let $u(x,y), c(x,y)$ and $a(x,y), D_iu(x,y)$ and $Du(x,y)$ be non-negative continuous functions for all $i=1,2$ defined for $x,y\in\mathbb{R}_+$ and $\alpha,\beta\in C^1(\mathbb{R}_+,\mathbb{R}_+)$ be non-decreasing functions in each variable, with $\alpha(x)\geq x$ on $\mathbb{R}_+$, and $\beta(y)\geq y$ on $\mathbb{R}_+$. Let $c(x,y)$ be non-decreasing in each variable $x,y\in\mathbb{R}_+$, and

$$\lim_{x\to\infty}u(x,y)=\lim_{x\to\infty}u(x,y)=0.$$

If

$$Du(x,y)\leq c(x,y)+\int_{\alpha(x)}^{\infty}\int_{\beta(y)}^{\infty}a(s,t)[u(s,t)+Du(s,t)]dsdt, \tag{2.1}$$

for all $x,y\in\mathbb{R}_+$, then

$$Du(x,y)\leq c(x,y)\int_{\alpha(x)}^{\infty}\int_{\beta(y)}^{\infty}a(s,t)exp\left[\int_s^{\infty}\int_t^{\infty}[a(\tau,\sigma)]d\tau d\right. \tag{2.2}$$

For all $x,y\in\mathbb{R}_+$.

Proof: Fix any $X,Y\in\mathbb{R}_+$. Then, for $x\leq X$ and $y\leq Y$, we have

$Du(x,y) \le c(X,Y) + \int_{\alpha(x)}^{\infty}\int_{\beta(y)}^{\infty} a(s,t)[u(s,t) + Du(s,t)]dsdt,$

Define a function $z(x,y)$ by

$$z(x,y) = c(X,Y) + \int_{\alpha(x)}^{\infty}\int_{\beta(y)}^{\infty} a(s,t)[u(s,t) + Du(s,t)]dsdt, \quad (2.3)$$

Then

$$\lim_{x\to\infty} z(x,y) = \lim_{x\to\infty} z(x,y) = c(X,Y),$$

And

$$Du(x,y) \le z(x,y). \quad (2.4)$$

By differentiating (2.3)

$$Dz(x,y) \le a(\alpha(x),\beta(y))[u(\alpha(x),\beta(y)) + Du(\alpha(x),\beta(y))]\alpha'(x)\beta'(y) \quad (2.5)$$

By integrating both sides of (2.4)

$$Du(x,y) \le \int_{\alpha(x)}^{\infty}\int_{\beta(y)}^{\infty} z(s,t)dsdt, \quad (2.6)$$

Now, using (2.4) and (2.6) in (2.5), we get

$$Dz(x,y) \le a(\alpha(x),\beta(y))\left[z(x,y) + \int_{\alpha(x)}^{\infty}\int_{\beta(y)}^{\infty} z(s,t)dsdt\right]\alpha'(x)\beta'(y). \quad (2.7)$$

If we put

$$v(x,y) = z(x,y) + \int_{\alpha(x)}^{\infty}\int_{\beta(y)}^{\infty} z(s,t)dsdt\,, \quad (2.8)$$

then

$$\lim_{x\to\infty} z(x,y) = \lim_{y\to\infty} z(x,y) = c(X,Y),$$

and

$$Dv(x,y) \le Dz(x,y) + z(x,y)\alpha'(x)\beta'(y).$$

using the fact that

$$Dz(x,y) \le a(\alpha(x),\beta(y))v(x,y)\alpha'(x)\beta'(y)$$

from (2.7) form (2.8) we have

$$Dv(x,y) \le [1 + a(x,y)]v(x,y)\alpha'(x)\beta'(y).$$

It is easy to estimate $v(x,y)$ by

$$v(x,y) \le c(X,Y)\int_{\alpha(x)}^{\infty}\int_{\beta(y)}^{\infty} [1 + a(s,t)]dsdt\,. \quad (2.9)$$

By substituting (2.9) in (2.7) and integrating both sides, we get

$$z(x,y) \le c(X,Y)\int_{\alpha(x)}^{\infty}\int_{\beta(y)}^{\infty} a(s,t)exp\left[\int_{s}^{\infty}\int_{t}^{\infty}[a(\tau,\sigma)]d\tau d\sigma,\right]dsdt\,.$$

Since X and Y are arbitraries and by substituting the value of $z(x,y)$ in (2.4), we obtain the inequality (2.2).

Remark 1 If we put $\infty = 0, \alpha(x) = x, \beta(y) = y$, and $c(x,y) = c_1(x) + c_2(y)$ in Theorem 2 we obtain Theorem 1 in [7].

Corollary 3. Let $u(x,y), c(x,y)$ and $a(x,y), D_i u(x,y)$ and $Du(x,y)$ be non-negative continuous functions for all $i = 1,2$ defined for $x,y \in \mathbb{R}_+$ and $\alpha,\beta \in C^1(\mathbb{R}_+,\mathbb{R}_+)$ be non-decreasing functions in each variable, with $\alpha(x) \ge x$ on $\mathbb{R}_+$, and $\beta(y) \ge y$ on $\mathbb{R}_+$. Let $c(x,y)$ be non-decreasing in each variable $x,y \in \mathbb{R}_+$, and

$$\lim_{x\to\infty} u(x,y) = \lim_{x\to\infty} u(x,y) = 0,$$

If

$$Du(x,y) \le c(x,y) + M\left[u(x,y) + \int_{\alpha(x)}^{\infty}\int_{\beta(y)}^{\infty} a(s,t)[u(s,t) + Du(s,t)]dsdt\right], \quad (2.10)$$

for all $x,y \in \mathbb{R}_+$, where $M > 0$ is constant, then

$$Du(x,y) \le c(x,y)exp\int_{\alpha(x)}^{\infty}\int_{\beta(y)}^{\infty}[M + Ma(s,t) + a(s,t)]dsdt, \quad (2.11)$$

for all $x,y \in \mathbb{R}_+$.

Proof : Fix any $X,Y \in \mathbb{R}_+$. Then, for $x \le X$ and $y \le Y$, then from (2.10)

$$Du(x,y) \le c(X,Y) + M\left[u(x,y) + \int_{\alpha(x)}^{\infty}\int_{\beta(y)}^{\infty} a(s,t)[u(s,t) + Du(s,t)]dsdt\right],$$

Define a function $z(x,y)$ by

$$z(x,y) = c(X,Y) + M\left[u(x,y) + \int_{\alpha(x)}^{\infty}\int_{\beta(y)}^{\infty} a(s,t)[u(s,t) + Du(s,t)]dsdt\right], \quad (2.12)$$

Then

$$\lim_{x\to\infty} z(x,y) = \lim_{x\to\infty} z(x,y) = c(X,Y)$$

And

$$Du(x,y) \le z(x,y). \quad (2.13)$$

By differentiating (2.12) and using (2.13), we have

$$Dz(x,y) \le z(x,y)[M + Ma(\alpha(x),\beta(y)) + a(\alpha(x),\beta(y))]\alpha'(x)\beta'(y),$$

Therefore

$z(x,y) \le c(X,Y)exp\int_{\alpha(x)}^{\infty}\int_{\beta(y)}^{\infty}[M + Ma(s,t) + a(s,t)]dsdt,$

Since X and Y are arbitraries and by substituting the value of $z(x,y)$ in (2.13), we obtain the inequality (2.11).

Remark 2. If we put $\infty = \mathbf{0}, \boldsymbol{\alpha(x) = x}, \boldsymbol{\beta(y) = y}$, and $\boldsymbol{c(x,y) = c_1(x) + c_2(y)}$ in corollary 3 we obtain theorem 2 in [8].

Corollary 4. Let $u(x,y), c(x,y)$ and $a(x,y), D_i u(x,y)$ and $Du(x,y)$ be non-negative continuous functions for all $i = 1,2$ defined for $x, y \in \mathbb{R}_+$ and $\alpha, \beta \in C^1(\mathbb{R}_+, \mathbb{R}_+)$ be non-decreasing functions in each variable, with $\alpha(x) \geq x$ on $\mathbb{R}_+$, and $\beta(y) \geq y$ on $\mathbb{R}_+$. Let $c(x,y)$ be non-decreasing in each variable $x, y \in \mathbb{R}_+$, and

$$\lim_{x\to\infty} u(x,y) = \lim_{x\to\infty} u(x,y) = 0,$$

If

$$Du(x,y) \leq c(x,y) + M\Big[u(x,y) + \int_{\alpha(x)}^{\infty}\int_{\beta(y)}^{\infty} a(s,t) Du(s,t)\,dsdt\Big],$$

for all $x, y \in \mathbb{R}_+$, where $M > 0$ is constant, then

$$Du(x,y) \leq c(x,y) exp \int_{\alpha(x)}^{\infty}\int_{\beta(y)}^{\infty} M[a(s,t)+1]\,dsdt,$$

for all $x, y \in \mathbb{R}_+$.

Proof : The proof of this Corollary follows the same arguments as in Corollary 3.

Remark 3. If we put $\infty = 0, \alpha(x) = x, \beta(y) = y$, and $c(x,y) = c_1(x) + c_2(y)$ in Corollary 4 we obtain the result in [12].

Theorem 5. Let $u(x,y), c(x,y)$ and $a(x,y), b(x,y)$ be non-negative continuous functions defined for $x, y \in \mathbb{R}_+$ and $\alpha, \beta \in C^1(\mathbb{R}_+, \mathbb{R}_+)$ be non-decreasing functions in each variable, with $\alpha(x) \geq x$ on $\mathbb{R}_+$, and $\beta(y) \geq y$ on $\mathbb{R}_+$. Let $c(x,y)$ be non-decreasing in each variable $x, y \in \mathbb{R}_+$. If

$$u(x,y) \leq c(x,y) + \int_{\alpha(x)}^{\infty}\int_{\beta(y)}^{\infty} a(s,t)u(s,t)\,dsdt \int_{\alpha(x)}^{\infty}\int_{\beta(y)}^{\infty} a(s,t)\Big[\int_s^{\infty}\int_t^{\infty} b(\tau,\sigma)u(\tau,\sigma)\,d\tau d\sigma\Big]ds\epsilon \quad (2.14)$$

For all $x, y \in \mathbb{R}_+$, then

$$u(x,y) \leq c(x,y) exp \int_{\alpha(x)}^{\infty}\int_{\beta(y)}^{\infty} a(s,t)\,dsdt + \int_{\alpha(x)}^{\infty}\int_{\beta(y)}^{\infty} a(s,t)\Big[\int_s^{\infty}\int_t^{\infty} b(\tau,\sigma)\,d\tau d\sigma\Big]dsdt, \quad (2.15)$$

For all $x, y \in \mathbb{R}_+$.

Proof: Since $c(x,y)$ is non-negative and non-decreasing, from (2.14) we have

$$\frac{u(x,y)}{c(x,y)} \leq 1 + \int_{\alpha(x)}^{\infty}\int_{\beta(y)}^{\infty} a(s,t)\frac{u(s,t)}{c(s,t)}\,dsdt + \int_{\alpha(x)}^{\infty}\int_{\beta(y)}^{\infty} a(s,t)\Big[\int_s^{\infty}\int_t^{\infty} b(\tau,\sigma)\frac{u(\tau,\sigma)}{c(\tau,\sigma)}\,d\tau d\sigma\Big]dsdt,$$

Define a function $z(x,y)$ by the right side of the last inequality. Then $z(x,y) \geq 0$,

$$\lim_{x\to\infty} z(x,y) = \lim_{x\to\infty} z(x,y) = 1, \frac{u(x,y)}{c(x,y)} \leq z(x,y),$$

and

$$Dz(x,y) \leq z(x,y)\Big[a(x,y) + a(x,y)\int_{\alpha(x)}^{\infty}\int_{\beta(y)}^{\infty} b(s,t)\,dsdt\Big]\alpha'(x)\beta'(y).$$

i.e

$$\frac{Dz(x,y).z(x,y)}{z^2(x,y)} - \frac{D_1 z(x,y) D_2 z(x,y)}{z^2(x,y)} \leq \left[a(x,y) + a(x,y)\int_{\alpha(x)}^{\infty}\int_{\beta(y)}^{\infty} b(s,t)\,dsdt\right]\alpha'(x)\beta'(y).$$

Thus

$$D_2\left[\frac{D_1 z(x,y)}{z(x,y)}\right] \leq \Big[a(x,y) + a(x,y)\int_{\alpha(x)}^{\infty}\int_{\beta(y)}^{\infty} b(s,t)\,dsdt\Big]\alpha'(x)\beta'(\quad (2.16)$$

By keeping y fixed, setting $x = s$, and integrating from $\alpha(x)$ to ∞ in (2.16), and again by keeping x fixed, setting $y = t$, and integrating from $\beta(y)$ to ∞ in the resulting inequality, we have

$$z(x,y) \leq c(x,y) exp \int_{\alpha(x)}^{\infty}\int_{\beta(y)}^{\infty} a(s,t)\,dsdt + \int_{\alpha(x)}^{\infty}\int_{\beta(y)}^{\infty} a(s,t)\Big[\int_s^{\infty}\int_t^{\infty} b(\tau,\sigma)\,d\tau d\sigma\Big]dsdt.$$

Finally, since

$$\frac{u(x,y)}{c(x,y)} \leq z(x,y)$$

We obtain the inequality (2.15).

Remark 4.

1. If we put $\infty = 0, \alpha(x) = x, \beta(y) = y$, and $c(x,y) = c_1(x) + c_2(y)$ in theorem 5 we obtain theorem 3 |7].
2. In the particular case when $b(x,y) = 0$, then the bound obtained in [8] reduces to :

$$u(x,y) \leq c(x,y) exp \int_{\alpha(x)}^{\infty}\int_{\beta(y)}^{\infty} a(s,t)\,dsdt. \quad (2.17)$$

Theorem 6. Let $u(x,y), c(x,y)$ and $a(x,y), b(x,y), f(x,y), D_i u(x,y)$ and $Du(x,y)$ be non-negative continuous functions for all $i = 1,2$ defined for $x, y \in \mathbb{R}_+$ and $\alpha, \beta \in C^1(\mathbb{R}_+, \mathbb{R}_+)$ be non-decreasing functions in each variable, with $\alpha(x) \geq x$ on $\mathbb{R}_+$, and $\beta(y) \geq y$ on $\mathbb{R}_+$.

And

$$\lim_{x\to\infty} u(x,y) = \lim_{x\to\infty} u(x,y) = 0$$

Let $K\big(u(x,y)\big)$ be a real-valued, positive, continuous, strictly non-decreasing, sub-additive, and sub-multiplicative function for $u(x,y) \geq 0$, and $H\big(u(x,y)\big)$ be a real-valued, positive, continuous and non-decrasing function defined for $x, y \in \mathbb{R}_+$. Assume that $c(x,y)$ and $f(x,y)$ are non-decreasing in each of the variables $x, y \in \mathbb{R}_+$. If

$$Du(x,y) \leq c(x,y) + f(x,y)H\left(\int_{\alpha(x)}^{\infty}\int_{\beta(y)}^{\infty} a(s,t)K\big(u(s,t)\big)dsdt\right) + \int_{\alpha(x)}^{\infty}\int_{\beta(y)}^{\infty} b(s,t)Du(s,t)dsdt\,, \tag{2.18}$$

for all $x, y \in \mathbb{R}_+$, then

$$u(x,y) \leq c(x,y) + f(x,y)H\left(G^{-1}\left[G(\xi) + \int_{\alpha(x)}^{\infty}\int_{\beta(y)}^{\infty} a(s,t)K\big(f(s,t)p(s,t)\big)dsdt\right]\right)p(x,y \tag{2.19}$$

for all $x, y \in \mathbb{R}_+$, where

$$p(x,y) = \int_{\alpha(x)}^{\infty}\int_{\beta(y)}^{\infty} exp\left[\int_s^{\infty}\int_t^{\infty} b(\tau,\sigma)d\tau d\sigma\right]dsdt\,. \tag{2.20}$$

$$\xi = \int_0^{\infty}\int_0^{\infty} a(s,t)K\big(c(s,t)p(s,t)\big)dsdt\,. \tag{2.21}$$

$$G(r) = \int_r^{\infty} \frac{ds}{K\big(H(s)\big)}. \tag{2.22}$$

Where G^{-1} is the inverse function of G, and

$$G(\xi) + \int_{\alpha(x)}^{\infty}\int_{\beta(y)}^{\infty} a(s,t)K\big(f(s,t)p(s,t)\big)dsdt \in dom(G^{-1})$$

for all $x, y \in \mathbb{R}_+$.

Proof : Define a function $z(x,y)$ by

$$z(x,y) = c(x,y) + f(x,y)H\left(\int_{\alpha(x)}^{\infty}\int_{\beta(y)}^{\infty} a(s,t)K\big(u(s,t)\big)dsdt\right), \tag{2.23}$$

then from (2.18), we have

$$Du(x,y) \leq z(x,y) + \int_{\alpha(x)}^{\infty}\int_{\beta(y)}^{\infty} b(s,t)Du(s,t)dsdt\,. \tag{2.24}$$

Clearly, $z(x,y)$ is a positive, continuous, and decreasing function in each of the variables $x, y \in \mathbb{R}_+$. Using (2.17) from Theorem 5 in (2.24), we get

$$Du(x,y) \leq z(x,y)exp\int_{\alpha(x)}^{\infty}\int_{\beta(y)}^{\infty} b(s,t)dsdt\,. \tag{2.25}$$

By integration, first with respect to x from x to ∞, and then with respect to y from y to ∞in the last inequality, we obtain

$$u(x,y) \leq z(x,y)p(x,y). \tag{2.26}$$

where $p(x,y)$ is defined in (2.20). From (2.23)we have

$$z(x,y) = c(x,y) + f(x,y)H\big(v(x,y)\big), \tag{2.27}$$

where

$$v(x,y) = \int_{\alpha(x)}^{\infty}\int_{\beta(y)}^{\infty} a(s,t)K\big(u(s,t)\big)dsdt\,. \tag{2.28}$$

Now, using (2.27) in (2.26)we get

$$u(x,y) \leq \big[c(x,y) + f(x,y)H\big(v(x,y)\big)\big]p(x,y) \tag{2.29}$$

From (2.28) and (2.29) and since K is a sub-additive and sub-multiplicative function, we obtain

$$v(x,y) \leq \int_{\alpha(x)}^{\infty}\int_{\beta(y)}^{\infty} a(s,t)K\Big(\big[c(s,t) + f(s,t)H\big(v(s,t)\big)\big]p(s,t)\Big)dsdt \leq \int_{\alpha(x)}^{\infty}\int_{\beta(y)}^{\infty} a(s,t)K\big(c(s,t)p(s,t)\big)dsdt + \int_{\alpha(x)}^{\infty}\int_{\beta(y)}^{\infty} a(s,t)K\Big(f(s,t)H\big(v(s,t)\big)p(s,t)\Big)dsdt\,.$$

Therefore

$$v(x,y) \leq \int_0^{\infty}\int_0^{\infty} a(s,t)K\big(c(s,t)p(s,t)\big)dsdt + \int_{\alpha(x)}^{\infty}\int_{\beta(y)}^{\infty} a(s,t)K\big(f(s,t)p(s,t)\big)K\Big(H\big(v(s,t)\big)\Big)dsdt\,.$$

Define a function $\Phi(x,y)$ by

$$\Phi(x,y) = \int_0^{\infty}\int_0^{\infty} a(s,t)K\big(c(s,t)p(s,t)\big)dsdt + \int_{\alpha(x)}^{\infty}\int_{\beta(y)}^{\infty} a(s,t)K\big(f(s,t)p(s,t)\big)K\Big(H\big(v(s,t)\big)\Big)dsdt\,. \tag{2.30}$$

Then

$$\lim_{x\to\infty}\Phi(x,y) = \lim_{y\to\infty}\Phi(x,y) = \int_0^{\infty}\int_0^{\infty} a(s,t)K\big(c(s,t)p(s,t)\big)dsdt = \xi. \tag{2.31}$$

and

$$\mathrm{v}(x,y) \leq \Phi(x,y). \tag{2.32}$$

Clearly, $\Phi(x,y)$ is a positive and decreasing function in y, then

$$D_1\Phi(x,y) = -\int_{\beta(y)}^{\infty} a(\alpha(x),t)K\big(f(\alpha(x),t)p(\alpha(x),t)\big)K\Big(H\big(v(\alpha(x),t)\big)\Big)dsdt\ \alpha'(x)$$
$$\geq -K\Big(H\big(\Phi(x,y)\big)\Big)\int_{\beta(y)}^{\infty} a(\alpha(x),t)K\big(f(\alpha(x),t)p(\alpha(x),t)\big)dsdt\ \alpha'(x).$$

i.e

$$\frac{D_1\Phi(x,y)}{K\left(H\big(\Phi(x,y)\big)\right)} \geq -\int_{\beta(y)}^{\infty} a(\alpha(x),t)K\big(f(\alpha(x),t)p(\alpha(x),t)\big)dsd \quad (2.33)$$

From (2.22) we have

$$D_1\mathrm{G}\big(\Phi(x,y)\big) = \frac{D_1\Phi(x,y)}{K\left(H\big(\Phi(x,y)\big)\right)} \geq -\int_{\beta(y)}^{\infty} a(\alpha(x),t)K\big(f(\alpha(x),t)p(\alpha(x),t)\big)dsdt \quad (2.34$$

Now, by setting $x = s$ and integrating from x to ∞ in (2.34), and using (2.31) we get

$$\Phi(x,y) \leq G^{-1}\left[G(\xi) + \int_{\alpha(x)}^{\infty}\int_{\beta(y)}^{\infty} a(s,t)K\big(f(s,t)p(s,t)\big)dsdt\right], \quad (2.35)$$

Finally, by substituting (2.27), (2.32) and (2.25), (2.35) we obtain the inequality (2.19).

Remark 5.

1. From the inequalities (2.29), (2.32) and (2.35) in the proof of Theorem 6 we can find this inequality

$$\mathrm{u}(x,y) \leq \mathrm{c}(x,y) + \mathrm{f}(x,y)H\left(G^{-1}\left[G(\xi) + \int_{\alpha(x)}^{\infty}\int_{\beta(y)}^{\infty} a(s,t)K\big(f(s,t)p(s,t)\big)dsdt\right]\right)\mathrm{p}(x,y).$$

2. If we put $\infty = 0, \alpha(x) = x, \beta(y) = y$, and $c(x,y) = c_1(x) + c_2(y), \mathrm{f}(x,y) = 1, \mathrm{H}(x) = \mathrm{K}(x) = x$ in theorem 6 we obtain Theorem 1 [12].

Corollary 7. Let $u(x,y), c(x,y)$ and $a(x,y), b(x,y), D_i u(x,y)$ and $Du(x,y)$ be non-negative continuous functions for all $i = 1,2$ defined for $x, y \in \mathbb{R}_+$ and $\alpha, \beta \in C^1(\mathbb{R}_+, \mathbb{R}_+)$ be non-decreasing functions in each variable, with $\alpha(x) \geq x$ on $\mathbb{R}_+$, and $\beta(y) \geq y$ on $\mathbb{R}_+$.

And

$$\lim_{x\to\infty} u(x,y) = \lim_{x\to\infty} u(x,y) = 0$$

Let $K\big(u(x,y)\big)$ be a real-valued, positive, continuous, strictly non-decreasing, sub-additive, and sub-multiplicative function for $u(x,y) \geq 0$. Assume that $c(x,y)$ is non-decreasing in each of the variables $x, y \in \mathbb{R}_+$. If

$$Du(x,y) \leq c(x,y) + \int_{\alpha(x)}^{\infty}\int_{\beta(y)}^{\infty} a(s,t)K\big(u(s,t)\big)dsdt + \int_{\alpha(x)}^{\infty}\int_{\beta(y)}^{\infty} b(s,t)Du(s,t)dsdt, \quad (2.36)$$

for all $x, y \in \mathbb{R}_+$, then

$$u(x,y) \leq c(x,y) + H\left(T^{-1}\left[T(\xi) + \int_{\alpha(x)}^{\infty}\int_{\beta(y)}^{\infty} a(s,t)K\big(p(s,t)\big)dsdt\right]\right)p(x,y), \quad (2.37)$$

for all $x, y \in \mathbb{R}_+$, where $p(x,y)$ and ξ are defined in theorem 6.

$$T(r) = \int_r^{\infty} \frac{ds}{K(s)}.$$

Where T^{-1} is the inverse function of G, and

$$T(\xi) + \int_{\alpha(x)}^{\infty}\int_{\beta(y)}^{\infty} a(s,t)T\big(p(s,t)\big)dsdt \in dom(T^{-1})$$

for all $x, y \in \mathbb{R}_+$.

Proof : The proof of this Corollary follows the same arguments as in Theorem 6.

Remark 6.

1. If we put $\mathrm{f}(x,y) = 1, \mathrm{H}(x) = x$ in theorem 6 then we obtain the result in Corollary 7.

2. If we put $\infty = 0, \alpha(x) = x, \beta(y) = y$, and $c(x,y) = c_1(x) + c_2(y), \mathrm{a}(x,y) = \mathrm{b}(x,y), \mathrm{K}(x) = x$ in corollary 7 we obtain theorem 1 in [7].

Corollary 8. Let $u(x,y)$, $a(x,y), b(x,y), D_i u(x,y)$ and $Du(x,y)$ be non-negative continuous functions for all $i = 1,2$ defined for $x, y \in \mathbb{R}_+$ and $\alpha, \beta \in C^1(\mathbb{R}_+, \mathbb{R}_+)$ be non-decreasing functions in each variable, with $\alpha(x) \geq x$ on $\mathbb{R}_+$, and $\beta(y) \geq y$ on $\mathbb{R}_+$.

And

$$\lim_{x\to\infty} u(x,y) = \lim_{x\to\infty} u(x,y) = 0.$$

If

$$Du(x,y) \leq M + \int_{\alpha(x)}^{\infty}\int_{\beta(y)}^{\infty} a(s,t)u(s,t)dsdt + \int_{\alpha(x)}^{\infty}\int_{\beta(y)}^{\infty} b(s,t)Du(s,t)dsdt, \quad (2.38)$$

for all $x, y \in \mathbb{R}_+$, where $M > 0$ is constant, then the following conclusions are true:

$$Du(x,y) \leq M\left(1 + \int_0^{\infty}\int_0^{\infty} a(s,t)p(s,t)dsdt \; exp\int_{\alpha(x)}^{\infty}\int_{\beta(y)}^{\infty} a(s,t)p(s,t)dsdt\right) exp\int_{\alpha(x)}^{\infty}\int_{\beta(y)}^{\infty} b(s,t)dsdt,$$

$$u(x,y) \leq M\left(1 + \int_0^{\infty}\int_0^{\infty} a(s,t)p(s,t)dsdt \; exp\int_{\alpha(x)}^{\infty}\int_{\beta(y)}^{\infty} a(s,t)p(s,t)dsdt\right)p(x,y)$$

For all $x, y \in \mathbb{R}_+$, where $p(x,y)$ and ξ are defined in theorem 6.

Proof : By setting $K(x) = x$ and $c(x,y) = M$ in Corollary 7, we obtain the results of this Corollary.

Corollary 9. Let $u(x,y), a(x,y), b(x,y), D_i u(x,y)$ and $Du(x,y)$ be non-negative continuous functions for all $i = 1,2$ defined for $x, y \in \mathbb{R}_+$ and $\alpha, \beta \in C^1(\mathbb{R}_+, \mathbb{R}_+)$ be non-decreasing functions in each variable, with $\alpha(x) \geq x$ on $\mathbb{R}_+$, and $\beta(y) \geq y$ on $\mathbb{R}_+$.

And

$$\lim_{x\to\infty} u(x,y) = \lim_{x\to\infty} u(x,y) = 0$$

Let $K\big(u(x,y)\big)$ be a real-valued, positive, continuous, strictly non-decreasing, sub-additive, and sub-multiplicative function for $u(x,y) \geq 0$. If

$$Du(x,y) \leq c_1(x) + c_2(y) + \int_{\alpha(x)}^{\infty}\int_{\beta(y)}^{\infty} a(s,t)K\big(u(s,t)\big)dsdt + \int_{\alpha(x)}^{\infty}\int_{\beta(y)}^{\infty} b(s,t)Du(s,t)dsdt\,,$$

For all $x, y \in \mathbb{R}_+$, where $c_1(x), c_2(y) > 0$, and $c'_1(x), c'_2(y) > 0$ are continuous functions defined for $x \geq 0, y \geq 0$ then

$$Du(x,y) \leq c_1(x) + c_2(y) + \left(T^{-1}\left[T(\xi) + \int_{\alpha(x)}^{\infty}\int_{\beta(y)}^{\infty} a(s,t)K\big(p(s,t)\big)dsdt\right]\right) exp \int_{\alpha(x)}^{\infty}\int_{\beta(y)}^{\infty} b(s,t)dsdt\,,$$

For all $x, y \in \mathbb{R}_+$, where

$$\xi = \int_{0}^{\infty}\int_{0}^{\infty} a(s,t)K\Big(\big(c_1(s) + c_2(t)\big)p(s,t)\Big)\,dsdt$$

And $p(x,y)$ and T are defined in corollary 7.

Proof : By setting $c(x,y) = c_1(x) + c_2(y)$ in Corollary 7 and using the same arguments in theorem 6, we obtain the results of this Corollary.

III. Retarded Non-Linear Integro-Differential Inequalities in n Independent Variables

In this section, we present some results of non-linear retarded integro-differential inequalities in n independent variables.

In what follows, for

$$x = (x_1, x_2, \dots x_n), t = (t_1, t_2, \dots t_n), \tilde{\infty} = (\infty, \infty, \dots, \infty),$$

We denote :

For $x, t \in \mathbb{R}_+^n$, we shall write $t \leq x$ whenever $t_i \leq x_i, i = 1,2,\dots,n$. For any $X = (X_1, X_2, \dots X_n) \in \mathbb{R}_+^n$, we shall write $x \leq X$ whenever $x_i \leq X_i, i = 1,2,\dots,n$.

$$\tilde{\alpha}(x) = \big(\alpha_1(x_1), \alpha_2(x_2), \dots \alpha_n(x_n)\big) \in C^1(\mathbb{R}_+^n, \mathbb{R}_+^n).$$

We denote $\hat{\alpha}(x) \geq x$ whenever $\alpha_i(x_i) \geq x_i$ for $i = 1,2,\dots,n$.

$$\int_{\tilde{\alpha}}^{\tilde{\infty}} dt = \int_{\alpha_1(x_1)}^{\infty}\int_{\alpha_2(x_2)}^{\infty} \dots \int_{\alpha_n(x_n)}^{\infty} dt_n \dots dt_2 dt_1.$$

Our main results are given in the following theorems.

Theorem 10. Let $u(x), a(x)$ and $c(x)$ be non-negative continuous functions defined for $x \in \mathbb{R}_+^n$ and $\tilde{\alpha} \in C^1(\mathbb{R}_+^n, \mathbb{R}_+^n)$ be non-decreasing functions in each variable, with $\tilde{\alpha}(x) \geq x$ on $\mathbb{R}_+^n$. Assume that $c(x)$ is non-decreasing in each variable $x \in \mathbb{R}_+^n$. if

$$u(x) \leq c(x) + \int_{\tilde{\alpha}(x)}^{\tilde{\infty}} a(t)u(t)dt, \tag{3.1}$$

for $x \in \mathbb{R}_+^n$, then

$$u(x) \leq c(x) exp \int_{\tilde{\alpha}(x)}^{\tilde{\infty}} a(t)dt. \tag{3.2}$$

Proof : Since $c(x)$ is non-negative and non-decreasing, from (3.1) we have

$$\frac{u(x)}{c(x)} \leq 1 + \int_{\tilde{\alpha}(x)}^{\tilde{\infty}} a(t)\frac{u(t)}{c(t)}dt$$

Define a function $z(x)$ by

$$z(x) = 1 + \int_{\tilde{\alpha}(x)}^{\tilde{\infty}} a(t)\frac{u(t)}{c(t)}dt,$$

then

$$z(x) > 0, \lim_{x_i\to\infty} z(x_1, \dots, x_n) = 1, i = 1,2,\dots,n, \frac{u(x)}{c(x)} \leq z(x)$$

$$\begin{cases} Dz(x) \leq a(x)z(x)\tilde{\alpha}'(x); if\ n\ is\ even \\ Dz(x) \geq -a(x)z(x)\tilde{\alpha}'(x); if\ n\ is\ odd \end{cases}$$

i.e

$$\frac{z(x)Dz(x)}{z^2(x)} - \frac{D_n z(x)\big(D_1 \dots D_{n-1}z(x)\big)}{z^2(x)} \leq a(x)\tilde{\alpha}'(x).$$

Therefore

$$\begin{cases} D_n\left(\dfrac{D_1 \dots D_{n-1}z(x)}{z^2(x)}\right) \leq a(x)\tilde{\alpha}'(x); if\ n\ is\ even \\ D_n\left(\dfrac{D_1 \dots D_{n-1}z(x)}{z(x)}\right) \geq -a(x)\tilde{\alpha}'(x); if\ n\ is\ odd \end{cases} \tag{3.3}$$

By integrating (3.3) with respect to x_n from x_n to ∞, we have

$$\begin{cases} -\left(\dfrac{D_1 \dots D_{n-1} z(x)}{z^2(x)}\right) \leq \displaystyle\int_{\alpha_n(x_n)}^{\infty} a(x_1, \dots, x_{n-1}, t_n) dt_n \alpha_1{}'(x_1) \dots \alpha_{n-1}{}'(x_{n-1})\,; if\ n\ is\ even \\ -\left(\dfrac{D_1 \dots D_{n-1} z(x)}{z^2(x)}\right) \geq -\displaystyle\int_{\alpha_n(x_n)}^{\infty} a(x_1, \dots, x_{n-1}, t_n) dt_n \alpha_1{}'(x_1) \dots \alpha_{n-1}{}'(x_{n-1})\,; if\ n\ is\ odd \end{cases},$$

thus

$$\begin{cases} \dfrac{D_1 \dots D_{n-1} z(x)}{z(x)} \geq D_{n-1}\left(\dfrac{D_1 \dots D_{n-2} z(x)}{z(x)}\right); if\ n\ is\ even \\ \dfrac{D_1 \dots D_{n-1} z(x)}{z(x)} \leq D_{n-1}\left(\dfrac{D_1 \dots D_{n-2} z(x)}{z(x)}\right); if\ n\ is\ odd \end{cases},$$

hence

$$\begin{cases} -D_{n-1}\left(\dfrac{D_1 \dots D_{n-2} z(x)}{z(x)}\right) \leq \displaystyle\int_{\alpha_n(x_n)}^{\infty} a(x_1, \dots, x_{n-1}, t_n) dt_n \alpha_1{}'(x_1) \dots \alpha_{n-1}{}'(x_{n-1})\,; if\ n\ is\ even \\ -D_{n-1}\left(\dfrac{D_1 \dots D_{n-2} z(x)}{z(x)}\right) \geq -\displaystyle\int_{\alpha_n(x_n)}^{\infty} a(x_1, \dots, x_{n-1}, t_n) dt_n \alpha_1{}'(x_1) \dots \alpha_{n-1}{}'(x_{n-1})\,; if\ n\ is\ odd \end{cases},$$

By integrating the last inequality with respect to x_{n-1} from x_{n-1} to ∞, we have

$$\begin{cases} \dfrac{D_1 \dots D_{n-2} z(x)}{z(x)} \leq \displaystyle\int_{\alpha_{n-1}(x_{n-1})}^{\infty} \int_{\alpha_n(x_n)}^{\infty} a(x_1, \dots, x_{n-2}, t_{n-1}, t_n) dt_n dt_{n-1} \alpha_1{}'(x_1) \dots \alpha_{n-2}{}'(x_{n-2})\,; if\ n\ is\ even \\ \dfrac{D_1 \dots D_{n-2} z(x)}{z(x)} \geq -\displaystyle\int_{\alpha_{n-1}(x_{n-1})}^{\infty} \int_{\alpha_n(x_n)}^{\infty} a(x_1, \dots, x_{n-2}, t_{n-1}, t_n) dt_n dt_{n-1} \alpha_1{}'(x_1) \dots \alpha_{n-2}{}'(x_{n-2})\,; if\ n\ is\ odd \end{cases},$$

By continuing this process, we get

$$\begin{cases} -\dfrac{D_1 z(x)}{z(x)} \leq \displaystyle\int_{\alpha_2(x_2)}^{\infty} \dots \int_{\alpha_n(x_n)}^{\infty} a(x_1, t_2, \dots, t_n) dt_n \dots dt_2 \alpha_1{}'(x_1)\,; if\ n\ is\ even \\ \dfrac{D_1 z(x)}{z(x)} \geq -\displaystyle\int_{\alpha_2(x_2)}^{\infty} \dots \int_{\alpha_n(x_n)}^{\infty} a(x_1, t_2, \dots, t_n) dt_n \dots dt_2 \alpha_1{}'(x_1)\,; if\ n\ is\ odd \end{cases}, \qquad (3.4)$$

By integrating (3.4) with respect to x_1 from x_1 to ∞, we have

$$z(x) \leq exp \int_{\tilde{\alpha}(x)}^{\tilde{\infty}} a(t) dt.$$

Finally, since $\frac{u(x)}{c(x)} \leq z(x)$ we obtain the inequality (3.2).

Remark 7. In the particular case when $n = 2$, $x \in \mathbb{R}_+^2$, $(\infty, \infty) = (0,0)$, $\alpha_1(x_1) = x_1, \alpha_2(x_2) = x_2$, and $c(x) = c_1(x_1) + c_2(x_2)$ then theorem 10 reduces to lemma 1 in [12].

Theorem 11. Let $u(x), c(x), a(x), D_i u(x)$ and $Du(x)$ be non-negative continuous functions for all $i = 1,2, \dots, n$ defined for $x \in \mathbb{R}_+^n$,

$$\lim_{x_i \to \infty} u(x_1, x_2, \dots, x_n) = 0, \forall i = 1,2, \dots, n.$$

Let $\tilde{\alpha} \in C^1(\mathbb{R}_+^n, \mathbb{R}_+^n)$ be non-decreasing functions in each variable, with $\tilde{\alpha}(x) \geq x$ on $\mathbb{R}_+^n$. Assume that $c(x)$ is non-decreasing in each variable $x \in \mathbb{R}_+^n$. if

$$Du(x) \leq c(x) + \int_{\tilde{\alpha}(x)}^{\tilde{\infty}} a(t)[u(t) + Du(t)] dt, \qquad (3.5)$$

for $x \in \mathbb{R}_+^n$, then

$$u(x) \le c(x)\left[1 + \int_{\tilde{\alpha}(x)}^{\tilde{\infty}} \left(a(t)exp\int_t^{\tilde{\infty}} \left(1 + a(\tau)\right)d\tau\right)dt\right]. \quad (3.6)$$

Proof: Fixe any $X \in \mathbb{R}_+^n$. Then, for $x \le X$ and from (3.5), we have

$$Du(x) \le c(X) + \int_{\tilde{\alpha}(x)}^{\tilde{\infty}} a(t)[u(t) + Du(t)]dt,$$

Define a function $z(x)$ by

$$z(x) = c(X) + \int_{\tilde{\alpha}(x)}^{\tilde{\infty}} a(t)[u(t) + Du(t)]dt, \quad (3.7)$$

Then

$$\lim_{x_i\to\infty} z(x_1, \dots, x_n) = c(X), i = 1,2, \dots, n,$$

$$Du(x) \le z(x), \quad (3.8)$$

By differentiating (3.8)

$$\begin{cases} Dz(x) \le a(x)[u(x) + Du(x)]\tilde{\alpha}'(x); if\ n\ is\ eve \\ Dz(x) \ge -a(x)[u(x) + Du(x)]\tilde{\alpha}'(x); if\ n\ is\ oc \end{cases} \quad (3.9)$$

By integrating both sides of (3.8)

$$u(x) \le \int_{\tilde{\alpha}(x)}^{\tilde{\infty}} z(t)dt, \quad (3.10)$$

Now, using (3.8) and (3.10) in (3.9) we get

$$\begin{cases} Dz(x) \le a(x)\left[z(x) + \int_{\tilde{\alpha}(x)}^{\tilde{\infty}} z(t)dt\right]\tilde{\alpha}'(x); if\ n \\ Dz(x) \ge -a(x)\left[z(x) + \int_{\tilde{\alpha}(x)}^{\tilde{\infty}} z(t)dt\right]\tilde{\alpha}'(x); if \end{cases} \quad (3.11)$$

If we put

$$v(x) = z(x) + \int_{\tilde{\alpha}(x)}^{\tilde{\infty}} z(t)dt, \quad (3.12)$$

Then

$$\lim_{x_i\to\infty} v(x_1, \dots, x_n) = c(X), i = 1,2, \dots, n,$$

And

$$\begin{cases} Dv(x) = Dz(x) + z(x)\tilde{\alpha}'(x); if\ n\ is\ even \\ Dv(x) = Dz(x) - z(x)\tilde{\alpha}'(x); if\ n\ is\ odd \end{cases}.$$

Using the fact that

$$\begin{cases} Dz(x) \le a(x)v(x)\tilde{\alpha}'(x); if\ n\ is\ even \\ Dz(x) \ge -a(x)v(x)\tilde{\alpha}'(x); if\ n\ is\ odd' \end{cases}$$

From (3.11) and $z(x) \le v(x)$ from (3.12), we have

$$\begin{cases} Dv(x) \le [1 + a(x)]v(x)\tilde{\alpha}'(x); if\ n\ is\ even \\ Dv(x) \ge -[1 + a(x)]v(x)\tilde{\alpha}'(x); if\ n\ is\ odd \end{cases}.$$

It is easy to estimate $v(x)$ by following the same arguments as in the proof of Theorem 10 as follows

$$v(x) \le c(X)exp\left[\int_{\tilde{\alpha}(x)}^{\tilde{\infty}} \left(1 + a(t)\right)dt\right]. \quad (3.13)$$

By substituting (3.13) in, (3.11) we get

$$\begin{cases} Dz(x) \le a(x)c(X)exp\left[\int_{\tilde{\alpha}(x)}^{\tilde{\infty}} \left(1 + a(t)\right)dt\right]\tilde{\alpha}'(x); if\ n\ is\ even \\ Dz(x) \ge -a(x)c(X)exp\left[\int_{\tilde{\alpha}(x)}^{\tilde{\infty}} \left(1 + a(t)\right)dt\right]\tilde{\alpha}'(x); if\ n\ is\ odd \end{cases}. \quad (3.14)$$

$$\lim_{x_n\to\infty} D_1 \dots D_{n-1}z(x_1, \dots, x_{n-1}, x_n) = 0.$$

By integrating (3.14) to x_n from x_n to ∞ , we have

$$\begin{cases} -D_1 \dots D_{n-1} z(x) \le c(X) \int_{\alpha_n(x_n)}^{\infty} a(x_1, \dots, x_{n-1}, t_n) \exp\left[\int_{\tau}^{\tilde{\infty}} \big(1 + a(\tau)\big) d\tau\right] \\ \qquad dt_n \alpha_1'(x_1) \dots \alpha_{n-1}'(x_{n-1}); \text{if } n \text{ is even} \\ -D_1 \dots D_{n-1} z(x) \ge -c(X) \int_{\alpha_n(x_n)}^{\infty} a(x_1, \dots, x_{n-1}, t_n) \exp\left[\int_{\tau}^{\tilde{\infty}} \big(1 + a(\tau)\big) d\tau\right] \\ \qquad dt_n \alpha_1'(x_1) \dots \alpha_{n-1}'(x_{n-1}); \text{if } n \text{ is odd} \end{cases}$$

By continuing this process, we obtain

$$\begin{cases} -D_1 z(x) \le c(X) \int_{\alpha_2(x_2)}^{\infty} \dots \int_{\alpha_n(x_n)}^{\infty} a(x_1, t_2, \dots, t_n) \exp\left[\int_{\tau}^{\tilde{\infty}} \big(1 + a(\tau)\big) d\tau\right] dt_n \dots dt_2 \alpha_1'(x_1) \,; \text{if } n \text{ is even} \\ D_1 z(x) \ge -c(X) \int_{\alpha_2(x_2)}^{\infty} \dots \int_{\alpha_n(x_n)}^{\infty} a(x_1, t_2, \dots, t_n) \exp\left[\int_{\tau}^{\tilde{\infty}} \big(1 + a(\tau)\big) d\tau\right] dt_n \dots dt_2 \alpha_1'(x_1) \,; \text{if } n \text{ is odd} \end{cases},$$

By integrating the last inequality with respect to x_1 from x_1 to ∞, we have

$$z(x) \le c(X) \exp \int_{\tilde{\alpha}(x)}^{\tilde{\infty}} a(t) \exp\left[\int_{\tau}^{\tilde{\infty}} \big(1 + a(\tau)\big) d\tau\right] dt.$$

Since X is arbitrary, by substituting the value of $z(x)$ in (3.8), we obtain the inequality (3.6).

Remark 8. In the particular case when $n = 2$, $x \in \mathbb{R}_+^2$, $(\infty, \infty) = (0,0)$, $\alpha_1(x_1) = x_1, \alpha_2(x_2) = x_2$,and $c(x) = c_1(x_1) + c_2(x_2)$ then theorem 11 reduces to Theorem 1 in [8]

Corollary 12. Let $u(x), c(x), a(x), D_i u(x)$ and $Du(x)$ be non-negative continuous functions for all $i = 1,2, \dots, n$ defined for $x \in \mathbb{R}_+^n$,

$$\lim_{x_i \to \infty} u(x_1, x_2, \dots, x_n) = 0, \forall i = 1,2, \dots, n.$$

Let $\tilde{\alpha} \in C^1(\mathbb{R}_+^n, \mathbb{R}_+^n)$ be non-decreasing functions in each variable, with $\tilde{\alpha}(x) \ge x$ on $\mathbb{R}_+^n$. Assume that $c(x)$ is non-decreasing in each variable $x \in \mathbb{R}_+^n$. if

$$Du(x) \le c(x) + M\left[u(t) + \int_{\tilde{\alpha}(x)}^{\tilde{\infty}} a(t)[u(t) + Du(t)]dt\right] \quad (3.15),$$

for $x \in \mathbb{R}_+^n$, then

$$Du(x) \le c(x) \exp \int_{\tilde{\alpha}(x)}^{\tilde{\infty}} [M + a(t) + Ma(t)]\, dt. \quad (3.16)$$

Proof : Fixe any $X \in \mathbb{R}_+^n$. Then, for $x \le X$ and from (3.15), we have

$$Du(x) \le c(X) + M\left[u(t) + \int_{\tilde{\alpha}(x)}^{\tilde{\infty}} a(t)[u(t) + Du(t)]dt\right],$$

Define a function $z(x)$ by

$$z(x) = c(X) + M\left[u(t) + \int_{\tilde{\alpha}(x)}^{\tilde{\infty}} a(t)[u(t) + Du(t)]dt\right], \quad (3.17)$$

then

$$\lim_{x_i \to \infty} z(x_1, \dots, x_n) = c(X), i = 1,2, \dots, n,$$

$$Du(x) \le z(x), \quad (3.18)$$

By differentiating (3.8)

$$\begin{cases} Dz(x) \le M[a(x) + Du(x)][u(x) + Du(x)]\tilde{\alpha}'(x); \text{if } n \text{ is even} \\ Dz(x) \ge -M[a(x) + Du(x)][u(x) + Du(x)]\tilde{\alpha}'(x); \text{if } n \text{ is odd} \end{cases}.$$

Using (3.18) and the fact that $Ma(x) \le z(x)$, we have

$$\begin{cases} Dz(x) \le [Ma(x) + a(x) + M]z(x)\tilde{\alpha}'(x); \text{if } n \text{ is even} \\ Dz(x) \ge -[Ma(x) + a(x) + M]z(x)\tilde{\alpha}'(x); \text{if } n \text{ is odd} \end{cases}$$

Therefore

$$z(x) \le c(X) \exp \int_{\tilde{\alpha}(x)}^{\tilde{\infty}} [M + a(t) + Ma(t)]\, dt.$$

Since X is arbitrary, by substituting the value of $z(x)$ in (3.18) we obtain the inequality (3.16).

Remark 9. In the particular case when $n = 2$, $x \in \mathbb{R}_+^2$, $(\infty, \infty) = (0,0)$, $\alpha_1(x_1) = x_1, \alpha_2(x_2) = x_2$,and $c(x) = c_1(x_1) + c_2(x_2)$ then corollary 12 reduces to theorem 2 in [7].

Theorem 13. Let $u(x), c(x), a(x), b(x), f(x), D_i u(x)$ and $Du(x)$ be non-negative continuous functions for all $i =$

$1,2,\ldots,n$ defined for $x\in\mathbb{R}_+^n$, and $\tilde{\alpha}\in C^1(\mathbb{R}_+^n,\mathbb{R}_+^n)$ be non-decreasing functions in each variable, with $\tilde{\alpha}(x)\geq x$ on $\mathbb{R}_+^n$.

$$\lim_{x_i\to\infty} u(x_1,x_2,\ldots,x_n)=0, \forall i=1,2,\ldots,n.$$

Let $K(u(x))$ be a real-valued, positive, continuous, strictly non-decreasing, sub-additive, and sub-multiplicative function for $u(x)\geq 0$, and $H(u(x))$ be a real-valued, positive, continuous and non-decrasing function defined for $x\in\mathbb{R}_+^n$. Assume that $c(x)$ and $f(x)$ are non-decreasing functions in each of the variables $x\in\mathbb{R}_+^n$. If

$$Du(x)\leq c(x)+f(x)H\left(\int_{\tilde{\alpha}(x)}^{\tilde{\infty}} a(t)K(u(t))dt\right)+\int_{\tilde{\alpha}(x)}^{\tilde{\infty}} b(t)Du(t)dt, \tag{3.19}$$

for all $x\in\mathbb{R}_+^n$, then

$$Du(x)\leq c(x)+f(x)H\left(G^{-1}\left[G(\xi)+\int_{\tilde{\alpha}(x)}^{\tilde{\infty}} a(t)K(f(t)p(t))dt\right]\right) exp \int_{\tilde{\alpha}(x)}^{\tilde{\infty}} b(t)dt, \tag{3.20}$$

for all $x\in\mathbb{R}_+^n$, where

$$p(x)=\int_{\tilde{\alpha}(x)}^{\tilde{\infty}}\left(exp\int_t^{\tilde{\infty}} b(\tau)d\tau\right)dt. \tag{3.21}$$

$$\xi=\int_{\tilde{0}}^{\tilde{\infty}} a(t)K(c(t)p(t))dt. \tag{3.22}$$

$$G(r)=\int_r^{\infty}\frac{ds}{K(H(s))}. \tag{3.23}$$

Where G^{-1} is the inverse function of G, and

$$G(\xi)+\int_{\tilde{\alpha}(x)}^{\tilde{\infty}} a(t)K(f(t)p(t))dt\in dom(G^{-1})$$

for all $x\in\mathbb{R}_+^n$.

Proof : Define a function $z(x)$ by

$$z(x)=c(x)+f(x)H\left(\int_{\tilde{\alpha}(x)}^{\tilde{\infty}} a(t)K(u(t))dt\right), \tag{3.24}$$

then from (3.19), we have

$$Du(x)\leq z(x)+\int_{\tilde{\alpha}(x)}^{\tilde{\infty}} b(t)Du(t)dt, \tag{3.25}$$

Clearly, $z(x)$ is a positive, continuous, and non-decreasing function in each of the variables $x\in\mathbb{R}_+^n$. Using Theorem 10 in (3.25), we get

$$Du(x)\leq z(x)exp\left(\int_{\tilde{\alpha}(x)}^{\tilde{\infty}} b(t)dt\right). \tag{3.26}$$

By integrating (3.26) with respect to x from x to $\tilde{\infty}$, we obtain

$$u(x)\leq z(x)p(x), \tag{3.27}$$

where $p(x)$ is defined in (3.21). From (3.24) we have

$$z(x)=c(x)+f(x)H(v(x)), \tag{3.28}$$

Where

$$v(x)=\int_{\tilde{\alpha}(x)}^{\tilde{\infty}} a(t)K(u(t))dt, \tag{3.29}$$

Now, using (3.28) in (3.27) we get

$$u(x)\leq [c(x)+f(x)H(v(x))]p(x), \tag{3.30}$$

From (3.29) and (3.30) and since K is a sub-additive and sub-multiplicative function, we obtain

$$v(x,y)\leq \int_{\tilde{\alpha}(x)}^{\tilde{\infty}} a(t)K\left([c(t)+f(t)H(v(t))]p(t)\right)dt \leq \int_{\tilde{\alpha}(x)}^{\tilde{\infty}} a(t)K(c(t)p(t))dt + \int_{\tilde{\alpha}(x)}^{\tilde{\infty}} a(t)K\left(f(t)H(v(t))p(t)\right)dt.$$

Therefore

$$v(x,y)\leq \int_{\tilde{0}}^{\tilde{\infty}} a(t)K(c(t)p(t))dt+\int_{\tilde{\alpha}(x)}^{\tilde{\infty}} a(t)K\left(f(t)H(v(t))p(t)\right)dt.$$

Define a function $\Phi(x)$ by

$$\Phi(x,y)=\int_{\tilde{0}}^{\tilde{\infty}} a(t)K(c(t)p(t))dt+\int_{\tilde{\alpha}(x)}^{\tilde{\infty}} a(t)K(f(t)p(t))K\left(H(v(t))\right)dt. \tag{3.31}$$

Then

$$\lim_{x_i\to\infty}\Phi(x)=\int_{\tilde{0}}^{\tilde{\infty}} a(t)K(c(t)p(t))dt=\xi. \tag{3.32}$$

And

$$v(x)\leq\Phi(x).$$

Clearly, $\Phi(x)$ is a positive and non-decreasing function in each variable $x_2,x_3,\ldots,x_n$, then

$$D_1\Phi(x)=-\int_{\alpha_2(x_2)}^{\infty}\ldots\int_{\alpha_n(x_n)}^{\infty} a(\alpha_1(x_1),t_2,\ldots,t_n)K(f(\alpha_1(x_1),t_2,\ldots,t_n)p(\alpha_1(x_1),t_2,\ldots,t_n)) K\left(H(v(\alpha_1(x_1),t_2,\ldots,t_n))\right)dt_2\ldots dt_n\alpha'_1(x_1),$$

hence

$$D_1\Phi(x) \geq -\int_{\alpha_2(x_2)}^{\infty} \dots \int_{\alpha_n(x_n)}^{\infty} a(\alpha_1(x_1), t_2, \dots, t_n)K\big(f(\alpha_1(x_1), t_2, \dots, t_n)p(\alpha_1(x_1), t_2, \dots, t_n)\big) K\Big(H\big(\Phi(x_1, t_2, \dots, t_n)\big)\Big)\, dt_2 \dots dt_n \alpha'_1(x_1)$$

$$D_1\Phi(x) \geq -K\Big(H\big(\Phi(x)\big)\Big)\int_{\alpha_2(x_2)}^{\infty} \dots \int_{\alpha_n(x_n)}^{\infty} a(\alpha_1(x_1), t_2, \dots, t_n)K\big(f(\alpha_1(x_1), t_2, \dots, t_n)p(\alpha_1(x_1), t_2, \dots, t_n)\big)\, dt_2 \dots dt_n \alpha'_1(x_1)$$

i.e

$$\frac{D_1\Phi(x)}{K\Big(H\big(\Phi(x)\big)\Big)} \geq -\int_{\alpha_2(x_2)}^{\infty} \dots \int_{\alpha_n(x_n)}^{\infty} a(\alpha_1(x_1), t_2, \dots, t_n)K\big(f(\alpha_1(x_1), t_2, \dots, t_n)p(\alpha_1(x_1), t_2, \dots, t_n)\big)\, dt_2 \dots dt_n \alpha'_1(x_1). \quad (3.33)$$

From (3.23) we have

$$D_1 \mathrm{G}\big(\Phi(x)\big) = \frac{D_1\Phi(x)}{K\Big(H\big(\Phi(x)\big)\Big)}. \quad (3.34)$$

$$D_1 \mathrm{G}\big(\Phi(x)\big) \geq -\int_{\alpha_2(x_2)}^{\infty} \dots \int_{\alpha_n(x_n)}^{\infty} a(\alpha_1(x_1), t_2, \dots, t_n)K\big(f(\alpha_1(x_1), t_2, \dots, t_n)p(\alpha_1(x_1), t_2, \dots, t_n)\big)\, dt_2 \dots dt_n \alpha'_1(x_1). \quad (3.35)$$

Now, by setting $x_1 = t$ and integrating from x_1 to ∞ in (3.35), and using (3.31) we get

$$\Phi(x) \leq G^{-1}\left[G(\xi) + \int_{\tilde{\alpha}(x)}^{\tilde{\infty}} a(t)K\big(f(t)p(t)\big)dt\right] \quad (3.36)$$

Finally, by substituting (3.28), (3.32) and (3.36), (3.34) we obtain the inequality (3.20).

Remark 10.
From the inequalities (3.30) and (3.36) in the proof of theorem 13, we can find this inequality

$$\mathrm{u}(x) \leq \mathrm{c}(x) + \mathrm{f}(x)H\left(G^{-1}\left[G(\xi) + \int_{\tilde{\alpha}(x)}^{\tilde{\infty}} a(t)K\big(f(t)p(t)\big)dt\right]\right)\mathrm{p}(x). \quad (3.37)$$

If we put $n = 2$, $x \in \mathbb{R}_+^2$, $(\infty, \infty) = (0,0), \alpha_1(x_1) = x_1, \alpha_2(x_2) = x_2$, and $c(x) = c_1(x_1) + c_2(x_2)$ $f(x) = 1, H(x) = K(x) = x$, and $a(x) = b(x)$ then Theorem 13 reduces to Theorem 1 in [7]

Corollary 14. Let $u(x), c(x), a(x), b(x), D_i u(x)$ and $Du(x)$ be non-negative continuous functions for all $i = 1,2, \dots, n$ defined for $x \in \mathbb{R}_+^n$, and $\tilde{\alpha} \in C^1(\mathbb{R}_+^n, \mathbb{R}_+^n)$ be non-decreasing functions in each variable, with $\tilde{\alpha}(x) \geq x$ on $\mathbb{R}_+^n$.

$$\lim_{x_i \to \infty} u(x_1, x_2, \dots, x_n) = 0, \forall i = 1,2, \dots, n.$$

Let $K\big(u(x)\big)$ be a real-valued, positive, continuous, strictly non-decreasing, sub-additive, and sub-multiplicative function for $u(x) \geq 0$, and $H\big(u(x)\big)$ be a real-valued, positive, continuous and non-decrasing function defined for $x \in \mathbb{R}_+^n$. Assume that $c(x)$ is non-decreasing function in each of the variables $x \in \mathbb{R}_+^n$. If

$$Du(x) \leq c(x) + \int_{\tilde{\alpha}(x)}^{\tilde{\infty}} a(t)K\big(u(t)\big)dt + \int_{\tilde{\alpha}(x)}^{\tilde{\infty}} b(t)Du(t)dt, \quad (3.38)$$

for all $x \in \mathbb{R}_+^n$, then

$$Du(x) \leq c(x) + \left(T^{-1}\left[T(\xi) + \int_{\tilde{\alpha}(x)}^{\tilde{\infty}} a(t)K\big(p(t)\big)dt\right]\right) exp \int_{\tilde{\alpha}(x)}^{\tilde{\infty}} b(t)dt,$$

for $x \in \mathbb{R}_+^n$, where where $p(t)$ and ξ are defined in theorem 13, and

$$T(r) = \int_r^{\infty} \frac{ds}{K(s)}.$$

Where T^{-1} is the inverse function of G, and

$$T(\xi) + \int_{\tilde{\alpha}(x)}^{\tilde{\infty}} a(t)K\big(p(t)\big)dt \in dom(T^{-1})$$

for $x \in \mathbb{R}_+^n$.

Proof: The proof of this Corollary follows the same arguments as in Theorem 13.

Remark 11. If we put $H(x) = x$ and $f(x) = 1$ in Theorem 13, then we obtain the result in corollary 14.

Corollary 15. Let $u(x), a(x), b(x), D_i u(x)$ and $Du(x)$ be non-negative continuous functions for all $i = 1,2, \dots, n$ defined for $x \in \mathbb{R}_+^n$, and $\tilde{\alpha} \in C^1(\mathbb{R}_+^n, \mathbb{R}_+^n)$ be non-decreasing functions in each variable, with $\tilde{\alpha}(x) \geq x$ on $\mathbb{R}_+^n$.

$$\lim_{x_i \to \infty} u(x_1, x_2, \dots, x_n) = 0, \forall i = 1,2, \dots, n.$$

If

$$Du(x) \leq M + \int_{\tilde{\alpha}(x)}^{\infty} a(t)u(t)dt + \int_{\tilde{\alpha}(x)}^{\infty} b(t)Du(t)dt,$$

for all $x \in \mathbb{R}_+^n$, where $M > 0$ is constant, then the following conclusion are true:

$$Du(x) \leq M\Big(1 + \int_0^{\infty} a(t)p(t)dt \, exp \int_{\tilde{\alpha}(x)}^{\infty} a(t)p(t)dt\Big) exp \int_{\tilde{\alpha}(x)}^{\infty} b(t)dt,$$

$$u(x) \leq M\left(1 + \int_0^{\infty} a(t)p(t)dt \, exp \int_{\tilde{\alpha}(x)}^{\infty} a(t)p(t)dt\right)p(x)$$

for $x \in \mathbb{R}_+^n$, where

$$p(x) = \int_{\tilde{\alpha}(x)}^{\infty} \left(exp \int_t^{\infty} b(\tau)d\tau\right)dt.$$

Proof: By setting $K(x) = x$ and $c(x) = M$in Corollary 14, we obtain the results of this Corollary.

Corollary 16. Let $u(x), a(x), b(x), D_i u(x)$ and $Du(x)$ be non-negative continuous functions for all $i = 1,2,\ldots,n$ defined for $x \in \mathbb{R}_+^n$, and $\tilde{\alpha} \in C^1(\mathbb{R}_+^n, \mathbb{R}_+^n)$ be non-decreasing functions in each variable, with $\tilde{\alpha}(x) \geq x$ on $\mathbb{R}_+^n$.

$$\lim_{x_i \to \infty} u(x_1, x_2, \ldots, x_n) = 0, \forall i = 1,2,\ldots,n.$$

Let $K(u(x))$ be a real-valued, positive, continuous, strictly non-decreasing, sub-additive, and sub-multiplicative function for $u(x) \geq 0$. If

$$Du(x) \leq \sum_{i=1}^n c_i(x_i) + \int_{\tilde{\alpha}(x)}^{\infty} a(t)K(u(t))dt + \int_{\tilde{\alpha}(x)}^{\infty} b(t)Du(t)dt,$$

for all $x \in \mathbb{R}_+^n$, where $c_i(x_i) > 0$ and $c'_i(x_i) \geq 0$ are continuous functions for $x_i \geq 0$ for all $i = 1,\ldots,n$ then

$$Du(x) \leq \sum_{i=1}^n c_i(x_i) + \Big(T^{-1}\Big[T(\xi) + \int_0^{\infty} a(t)K(p(t))dt \, exp \int_{\tilde{\alpha}(x)}^{\infty} a(t)K(p(t))dt\Big]\Big) exp \int_{\tilde{\alpha}(x)}^{\infty} b(t)dt,$$

for $x \in \mathbb{R}_+^n$, where where $p(t)$ and T are defined in corollary 14, and

$$\xi = \int_{\tilde{\alpha}(x)}^{\infty} a(t)K\left(p(t)\sum_{i=1}^n c_i(t_i)\right)dt.$$

Proof: By setting $c(x) = \sum_{i=1}^n c_i(x_i)$ in Corollary 14 and using the same arguments in [ ] and Theorem 13, we obtain the result of this Corollary.

REFERENCES

[1] Bainov, D., and P. Simeonov. Integral inequalities and applications. Kluwer Academic Publishers (1992).

[2] Denche, M. and Khellaf, H. Integral inequalities similar to Gronwall inequality. Electronic Journal of Diﬀerential Equations,176 (2007), 1-14.

[3] Khellaf, H. On integral inequalities for functions of several independent variables. Electronic Journal of Diﬀerential Equations,123 (2003), 1-12.

[4] Khellaf,H. Nonlinear delay integral inequalities for multi-variable functions. Electronic Journal of Diﬀerential Equations,169 (2011), 1-14.

[5] Oguntuase, J. A. On an inequality of Gronwall. Journal of Inequalities in Pure and Applied Mathematics,2(2001).

[6] Pachpatte, B. G. On some fundamental integrodiﬀerential and integral inequalities. An. Sci. Univ. Al. I. Cuza, Iasi,29(1977), 77–86.

[7] Pachpatte, B. G. On some new integrodiﬀerential inequalities of the Wendroff type. Journal of Mathematical Analysis and Applications,73(1980), 491–500.

[8] Pachpatte, B. G. Inequalities for diﬀerential and integral equations. Academic Press (1998).

[9] Pachpatte, B. G. On a certain retarded integral inequality and its applications. Journal of Inequalities in Pure and Applied Mathematics,5(2004).

[10] Snow, D. R. Gronwall's inequality for systems of partial diﬀerential equations in two independent variables. Proceedings of the American Mathematical Society,33(1972), 46-54.

[11] Yeh, C. C. and Shih, M. H. The Gronwall-Bellman inequality in several variables. Journal of Mathematical Analysis and Applications,86(1982), 157-167.

[12] Zhang, H. and Meng, F. Integral inequalities in two independent variables for retarded Volterra equations. Applied Mathematics and Computation,199(2008), 90–98.